DEFINING WITHIN A DYNAMIC GEOMETRY ENVIRONMENT: NOTES FROM THE CLASSROOM

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ABSTRACT. *This paper concerns the activity of defining. We report about an experiment in which we studied students' behavior in constructing and classifying quadrilaterals within a dynamic geometry environment (Cabri-Géomètre). In particular, we considered the problem of the consistence of certain definitions with the constructions made with Cabri, i.e. we used the microworld to make students reflect on the adequacy (within the microworld) of the definition they use. The findings show that there are kinds of thinking that are developed as a result of the interaction with the tool and suggest considerations on the problem of providing students with a meaningful and active approach to theoretical thinking.*

INTRODUCTION AND THEORETICAL FRAMEWORK

In this paper we discuss the problem of defining that we have faced elsewhere, see (Furinghetti & Paola, 1999 and 2000). The reason why we pay such a great attention to the theme of definition is that it is the first gate to enter a theory. Thus a way of defining not suitable to students' mind may affect the entire path in the construction of a theory. This theme is set in our general concern of providing students with means to approach theoretical thinking with *awareness*. Other authors have faced the issue of awareness in doing mathematics, e.g. Mason (1987). In using the term "awareness" we means that students must be active participants in the process of constructing a theory and have to grasp the meaning of what they are doing. This view is based both on recent literature in mathematics education and on evidence emerging from the past. At the beginning of the twentieth century, when the foundation movement stressed the importance of the axiomatic method and logic, mathematical programs (and, as a consequence, textbooks and teaching style) became oriented to follow a pattern based on axioms-theorems-deduction. This "axiomatic style" affected mathematics teaching in many countries. Even at the beginning, however, this approach was questioned by teachers, as evidenced by papers appearing in early journals oriented to mathematics teaching, and also by important mathematicians. Felix Klein claimed that in school, as well as in research, the phase of formalization must be preceded by a phase of exploration based on intuition. He wrote: "I maintain that mathematical intuition [...] is always far in advance of logical reasoning and covers a wider field". (Klein, 1896, p.246). We find a similar statement in the introduction of a school geometry book by a famous Italian mathematician, Francesco Severi. He wrote (1930, Vallecchi, Florence, p.IX; our translation): "Who is aware of the value of foundation theories, does not make the dangerous mistake of giving to the elementary teaching a critical and excessively abstract style." Giovanni Vailati (1907), an Italian secondary teacher and researcher in logic, supported a method of teaching in which exploration, experimental mathematics, drawing with rule and compass have to precede deduction.

In the 1920s the idea of a "genetic" principle took shape and interesting treatises were published[1]. For example, Gusev and Safuanov (2000) refer about a school geometry book by N. A. Izvolsky, in which it was advocated that teachers explain to students the origin of geometrical theorems. According to the author, when this is done, students see geometry in a different way. The idea of a genetic approach later took a definite form in the book by Otto Toeplitz (1963) on infinitesimal calculus. This author was aware that in this domain the notion that learning mathematics takes place in a sequence predetermined by mathematical logic has shown its pedagogical limitations. Indeed, when organized around their logical kernel, the definitions of the main concepts of calculus (integrals, limits, derivatives) come out of the blue and the burden of formal rules and of theorems makes it difficult for students to grasp the meaning of what they are doing.

Our historical outline stresses the widespread (over the years and across the countries) concern about providing students with an active and meaningful approach to theoretical thinking and about the search for mediators/environments to realize this approach. Some of the authors we have mentioned look at history as a mediator effective for this purpose. Others of them consider different mediators: in some passage of Izvolsky's writings we find an embryo of classroom discussion, Vailati's program is centered on the use of exploration and of mathematical instruments.

In the present paper we have chosen the dynamic geometry software *Cabri-Géomètre* as a mediator to make students aware in defining. Our aim is to study which strategies students apply and how this environment affects their behavior. To realize it we have studied the work on the definition of quadrilaterals of 21 students of junior high school (15 years old). Quadrilaterals (which, in theory, were a topic known by our students) have been chosen because the focus of the described activity is not on the mathematical object to be defined, but just on definitions considered as mathematical objects which need of careful reflection. We think one masters and reifies a concept when this concept is used as an object. For example, one understands the concept of function when the function becomes an element of the domain in which he/she works. Thus one starts to understand what a definition is when definition itself becomes an object of study[2].

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[1] A pioneer example of the application of this method is offered by the book *Éléments de géométrie* by the mathematician Alexis-Claude Clairaut first published in 1741 (Paris: David fils).

[2] For a wider description of our theoretical framework on definition see (Furinghetti & Paola, 1999 and 2000; Shir & Zaslavsky, 2001).

The family of quadrilaterals has been often considered for the purpose of studying the problem of defining (both as for students and for teachers). For example, among the articles and departments (about 700 and 1000 respectively) published in the years 1990-2000 in the journal *Mathematics teacher* we have singled out only 26 contributions that may be considered belonging to the domain of definitions. Of them, 6 treat quadrilaterals. The reason of the popularity of quadrilaterals as object of definition is that there is a great experience (from Euclid onwards) about this subject both from the theoretical and from the teaching side. As shown in the papers (De Villiers, 1994 and 1998; Shir & Zaslavsky, 2001) this subject allows to work on hierarchical classification and application of notions about sets, it may be treated through verbalization or visualization, and it is a good field in which one may test the notion of equivalent definitions, see (Leikin & Winicki-Landman, 2000). Moreover this subject is suitable to exploit the intrinsic cognitive character of the dynamic geometry environment, see (Jones, 1998 and 2000; Mariotti, 2000). In particular, we deal with the problem of the consistence of certain definitions with the constructions that can be done with Cabri. We use the microworld to make students reflect on the adequacy (within the microworld) of the definitions they use. Our study has not the pretense to give final answers to the issues in question, but we hope that it can be a further tessera in the mosaic that researchers are constructing about these issues.

METHODOLOGY

The experiment involved 21 Italian 10-grade students with previous experience of using Cabri for solving geometrical problems. They worked in pairs (one PC per pair), except one group of three students. The teacher and a researcher acting as observer (he was one of the authors) were present. The observer was not passive, but talked with students and addressed their activity to Cabri in order to obtain more information about the interaction student-Cabri. We know from other experiments that if the teacher use Cabri as a blackboard students do not enter completely in the "logic" of software, see (Arzarello et al., 1998).

The first day of the experiment the students received the following instructions:

- Give a classification of quadrilaterals.
- You can use what you know from the previous school years, and, if you think it is useful, Cabri.
- Please keep a diary of the sessions (the final classification, discussions with the mates working with you, ideas that have not been completely developed).
- Remember that a construction with Cabri is validated only if it is kept by the dragging test.

All students used Cabri; after two sessions of work they wrote a report, as it was requested. In the next session the teacher orchestrated a classroom discussion on the students' work, and wrote a report. Since she had the impression that the less extroverted students were not able to express their thoughts in front of all mates, she asked students to answer some written questions prepared by her. Our reflections are based on the notes by the observer, students' reports, the report of the teacher, and the written interviews.

Few days before the experiment students had classified triangles (only through paper and pencil), nevertheless they were quite worried about the task of classifying quadrilaterals. The reason of that was the **anxiety for the technical problems linked to Cabri**[3]. The presence of an external person, however, has motivated students and they worked with good will. The groups acted collaboratively. Who used the mouse was the leader in the activity, but the low number of students in each group allowed to use the mouse alternatively (at least this happened in the most collaborative groups), so that the risk of passivity in the group was avoided.

All groups began **with the construction of a square**. The most common sequence was "to draw **a circle** and a square inscribed or circumscribed". The pair Dalila-Chiara drew a circle and two points on it, then the square circumscribed to the circle. At the beginning they wrote the macro construction before checking if their construction passed the dragging test. They had four times the message "Macro is not consistent". They never checked the geometrical construction, but wrote again the macro until the computer said that it was correct (the construction, indeed, was correct, while the macro was wrong).

This behavior was widespread among students working with the dynamic software. Also the pairs Elisa-Michela and Paolo-Carlotta did not think to validate their construction when found that their rectangles did not pass the dragging test. This means that, even if they were able to construct macro and had sufficient abilities to use the software, they did not enter the logic of the tool. They worked only at the level Laborde (1998) terms "spatio-graphical level": thus they were not able to use Cabri as a mediator to pass from drawing to the geometrical theory.

The other groups (except one) made **drawings which were figural (i.e. not obeying geometrical rules)**. After having drawn a parallelogram as a mere drawing, Luca **asked himself** "How can I decide that this is really a parallelogram?" This is a question that would not have arisen without Cabri. It was spontaneous to **discuss on which conditions** the figure was really a parallelogram. It was important to have an expert at disposal (the teacher or the observer) who listened to the questions and answered in real time, not after some time when the questions are no more important.

Paolo and Carlotta drew a circle and after drew the symmetric C of a point B respect to the center O, then constructed the perpendicular line to the line BC. The intersection of this perpendicular line with the circle gave two points A and D. $ABCD$ was the resulting square. After that they used (correctly) symmetries to obtain a rhombus. They used **symmetries** and **definitions which are construction-oriented** (i.e. rich of hints for constructing). They succeeded in doing figure, even if they were

[3] We have emphasized with the boldface type the features we consider interesting in our experiment.

not able to state the **characteristic properties** of the figures they have correctly drawn. When the researcher asked a definition of the rhombus they repeated the construction learnt in junior secondary school based on drawing two perpendicular lines and taking congruent segments on them.

Elisa and Michela constructed a rectangle by combining two right-angled triangles. They tried with translations, with 180° degrees rotation. This pair was very collaborative, students tried to explain how it is possible to obtain rectangles from right-angled triangles, but the figures they produced did not pass the dragging test. Thon and Federica were much worse: they drew objects and erased them without the project necessary to perform the task in Cabri.

The attitude of combining blocks of elementary figures to obtain sophisticated figures, as it is done with the *Lego* blocks, was even more evident in the second session dedicated to trapezia. Students looked at the figure as a mere drawing and act as a scanner in reproducing the stereotyped figures of their textbook. The conceptual aspect was absent. In this behavior we may see a certain analogy with the behavior in algebra were signs are manipulated without meaning. Cabri may be useful to change this behavior, since emphasizes the construction-oriented aspects to the detriment of the pure reproductive ones and thus it calls for meaning.

From this first session we may draw some general conclusions that have been corroborated in the second session on trapezia:

- Students have in their mind figures (not the properties of figures).
- They desperately explore Cabri looking for a way to reproduce figures and use strange means and tricks on the Cabri screen. Figures they have in mind are static. They, indeed, recognize only the figures, which are in particular configurations. We'll find these observations also in the teacher's report. For students the construction is not validated by the fact that a certain property is kept by dragging, but by the drawing obtained in a particular position. Cabri has made visible the students' remoteness from the theory they should study.
- Students do not use the potentialities of the tool in a rational way.
- Only one pair uses symmetry. This is done to construct the figure, but not to define the figure.
- Some behavior observed is fostered by the current teaching style: since many years drawing with rule and compass has been abandoned in school.

THE TEACHER'S REPORT ON THE DISCUSSION

The teacher's report reflects her concern about the practical teaching problems she has to face. The report is based on the discussion in classroom. As an example of what happened during this session we give the following description of the discussion about "dragging".

Chiara began by stating that if the construction is correct, the shape of the figure is kept.

> Elisa: "No! Because sides shorten or lengthen!"
>
> Patrizia: "But what do you mean by shape of a figure? Even if measures change a square is always a square!"

This clam addressed the discussion on what the "shape" of figure is. For someone it was "the set of sides", for others "the construction one has made", "the aspect with which the figure appears", "enlargement or reduction (scaling off)". Ambra stated that perimeter and area of figures are kept by dragging. The teacher asked whether the dragging is an isometry; Ambra was puzzled and changed her mind. The teacher writes in her report that, even if isometries have been treated in the previous school year, students' reactions show that isometries remained very abstract. The movement originates confusion and students have difficulties in singling out invariants. This difficulty emerged also in the written interviews. Students were more able in recognizing the symmetric of a figure in respect to a point or a straight line and in recognizing symmetries of figures. Some student have noticed that the "little hand", which allows to drag, appears only in some positions.

ANSWERS TO THE WRITTEN INTERVIEW

The questions of the written interview have been prepared by the teacher on the ground of the crucial points emerged in the discussion. They concerned:

- which quadrilateral has been the starting figure
- what is kept and what is left by dragging
- how the passage from the construction to the definition happens: 1. Listing the actions in the menu? 2. Enunciating the properties used in the construction? 3. Enunciating only the properties sufficient to construct the figure?
- why it is required to drag the figure one has constructed.

Students answered individually on the ground of the work at the computer and the discussion in classroom. Protocols give interesting insights on kinds of thinking brought to light by the activity with Cabri. All students confirmed that they started from the square. A paradigmatic explanation of this fact given by a student is:

> I started from the square because it is the quadrilateral with more properties and because I can imagine it more easily than the other quadrilaterals.

Other students say that "The square is the easiest quadrilateral". The generic is more difficult to be conceived than the particular and figures with regularities (i.e. specified figures) are more easily perceived.

To have started from a square has reflections on the classification of quadrilaterals. Even if this subject had been developed before in classroom, 11 students did not answer. Six students drew a figure with Venn diagrams showing the hierarchical classification presented in their the textbook (may be reproduced in a ritual way). Four students chose a classification going from squares to trapezia, which is inspired by the sequence they used in constructing quadrilaterals with Cabri. Two of them simply listed quadrilaterals, one explicitly wrote that she did not agree with her textbook classification. One student drew the figure below ("quadrato" means "square"). The dynamic geometry environment, indeed, has oriented to a different criterion of classification, which we may term "by default". It is a kind of reverse hierarchy: one starts from the more specified figure (the square) having the greatest number of properties and goes on by leaving some properties.

CONCLUDING REMARKS

Coming back to the issue discussed in the introduction (what may be the suitable sequence for introducing students with awareness to theoretical thinking) we have now further elements of reflections. From many examples provided by ethnomathematics of the present and of the past we already know that each environment gives different stimuli and opportunities to mathematical thinking. As it is reported in (Gerdes, 1988) Mozambicans peasants conceive and use "spontaneously" properties of parallelograms, which in the teaching sequence of our schools are theorems that must be proved. In teaching spatial geometry we have observed that students have difficulties in dealing with some polyhedra (e.g. dodecahedron), but this does not mean that these polyhedra are difficult to be conceived in an absolute way. Artmann (2001) reports that not only dodecahedra appear naturally (this is one of the three shapes in which the mineral of pyrite crystallizes), but also bronze dodecahedra were popular craft objects in Roman Imperial times. In the same way the dynamic geometry environment, as a new kind of ethnomathematical feature, brings to light forms of approaching mathematical situations different from those emerging when working with paper and pencil (with or without rule and compass). In our experiment the dynamic geometry software enriched students' experience in certain fields, by providing new situations. Also it revealed itself a powerful environment to detect students' behaviors and difficulties. In addition, the discussion orchestrated by the teacher brought to light interesting teaching implications: the crossing of metric and non-metric properties, the fuzziness of the concept of transformation, the relation between perimeter, area and shape, the difficulty of invariants.

In our experiment we have observed a kind of "computer anxiety", i.e. students' main concern was to exploit the facilities of Cabri rather than to design a project for performing the task (constructing and classifying quadrilaterals). The choice of the circle as a figure on which to work for constructing, and of the square instead of a generic quadrilateral as a starting figure for classifying quadrilaterals is conditioned by the dynamic geometry software.

The environment of Cabri addresses students towards definitions construction-oriented. A student constructed rhombuses starting from the properties of diagonals. And, when he attempted to inscribe a given rhombus in the circle that he had drawn at the beginning, he discovered that this is possible on particular conditions. Thus a statement to be proved may be spontaneously generated by the activity with Cabri. Also the environment fosters the use (even limited) of symmetries, which are object of teaching, but are rarely used by students when working with paper and pencil.

We point out that the persistence of the figural conceptions and the reluctance to use the dragging test show the lack of links with theory. Sometimes students seem to consider the construction of figures as a mere practical activity separated from a mathematical theory. Educational research provides teachers with means to perceive the existence of this gap and to fill it with appropriate strategies.

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