PATTERNS OF REASONING

Marco Cartiglia*, Fulvia Furinghetti* & Domingo Paola**
*Dipartimento di Matematica dell'Università di Genova, Italy
**Liceo Scientifico 'A. Issel', Finale Ligure (Sv), Italy

ABSTRACT. I*n this paper we report on some patterns of reasoning, which emerged during an activity of proving, a mathematical statement performed by nine grade and university mathematics students. The statement in question involves drawing figures, working in arithmetic and in algebra. As for secondary students we detected fluency, flexibility and ability of verbalizing their reasoning. In particular, we will focus on the behavior of a student who through drawings succeeded in giving meaning to algebraic manipulation. The solutions of the university students were conditioned by the ballast of the formal style used in university course of mathematics.*

INTRODUCTION AND THEORETICAL FRAME

In works presented in PME meetings, see (Furinghetti & Paola, 2003), and in journals, see (Furinghetti, Olivero & Paola, 2001) we have focused on patterns of reasoning emerging when students are involved in activities of exploring, producing and validating conjectures. To rouse motivation and non-routine behaviors we set these activities in stimulating contexts such as group working, classroom discussion, and use of technology. In the present paper we go on in our investigation on patterns of reasoning by studying the results of an experiment in which proof was proposed as an intellectual challenge. In history this challenge has been fundamental for the production of new mathematical ideas and some authors, (De Villiers, 1996) for one, think that even nowadays it may be a motivation in classroom. The aim of the study is twofold:

- as researchers in math education to collect information about students' reasoning
- as teachers or teachers educators to outline some didactical implications.

In our investigation we had in mind certain aspects of students' reasoning to be analyzed, which guided the choice of the statements to be proved. In the following we briefly discuss these aspects.

One of the driving forces in performing mathematical tasks is *transformational reasoning*. According to (Simon, 1996) transformational reasoning is a third type of reasoning (beyond deduction and induction), which is not a mere gathering of information, but rather the development of a feeling for the mathematical situation a person is facing. It is the realization (physical or mental) of an operation or a set of operations on objects that brings to reconsider the transformations which the objects undergone to and the results of that operations. In transformational reasoning it is central the ability to consider not a static state, but rather a dynamic process in which a new state or a continuity of state are generated. Transformational reasoning is

reasoning by analogy and anticipation. It may produce a different way of thinking to mathematical objects, as well as a different set of questions and problems. Transformational reasoning is enhanced by fluency and flexibility, that is to say the abilities to overcome fixations in mathematical situations and to produce creative thinking within mathematical situations, see (Haylock, 1987). Gray & Tall (1994) focus on the flexibility as a mean for linking processes and concepts.

Among the abilities necessary to mathematical activities Selden and Selden (1995) take into consideration concepts reformulation. When a statement to be proved is given, the solver firstly needs to understand it. This may happen through reformulating the statement by paraphrase with words, by gestures, by figures, by symbols, by the production of examples. Moore (1994) acknowledges as proving ability the generation and the use of examples. Examples may work as prototypes, but have not to become stereotypes, see (Presmeg, 1992). The solver has to apply on examples a process of abstraction and generalization, that is, borrowing the expression from (Mason & Pimm, 1984) "seeing the general in the particular".

A student's behavior in proving maybe analyzed according to the framework of proof schemes defined by Harel and Sowder (1998) as what constitutes ascertaining and persuading for a student. The proof schemes are grouped in three main classes:

- external conviction (three types: ritual, authoritarian, symbolic)
- empirical (two types: inductive, perceptual)
- analytical (two types of proof schemes: transformational, axiomatic).

A ritual proof scheme manifests itself in the behavior of judging mathematical arguments only on the basis of their surface appearance: false arguments are accepted because they look like usual proofs, and on the contrary justifications that are even convincing are rejected because "they don't look as mathematical proofs". In a symbolic scheme mathematical facts are proved using only symbolic reasoning, i.e. using symbols without reference to their meaning. An authoritarian proof scheme relies on the authority of someone (book, teacher).

An inductive proof scheme relies on few examples without generalization. A perceptual proof scheme is based on rudimentary mental images without resorting to deduction.

A transformational proof scheme encompasses a deductive process including generality, goal-oriented and anticipatory mental operations, and transformational images. In addition to that an axiomatic proof scheme contemplates the presence of an axiomatic system.

According to Rodd (2000, p.231) the following questions are crucial: "(b) What is the personal nature of proof? […] Or why are students' personal justifications different from the paradigmatic mathematical proof? […], And (c) What might warranting mean in classroom practice?" These issues are discussed in (Hanna, 1990; Hersh, 1993) as for students and mathematicians, Barbin (1988) as for the historical point of view.

Our experiment took place in two different settings (setting **A**: secondary school, setting **B**: university). A set of problems including proof was given to students. The students knew that their performance would not be assessed with a mark. They were only asked to engage as much as possible in the solution of the problems and to write all their thoughts during the solution. We also asked them explicitly to write the difficulties encountered and if they enjoyed the problems. All students signed with a pseudonym their protocols. In the present paper we will focus on the following problem:

> Given a cube made up by little cubes all equal, take away a full column of little cubes. The number of the remaining little cubes is divisible by six. Try to explain why this happens.

It has had been chosen according to the following requirements:

- it is expressed by words
- it involves concepts that are at the grasp of the students
- it does not involves only rote manipulation, but rather requires to look at algebraic formulas with meaning and awareness
- even if the property to be proved is given, the form of the statement (including the invitation " Try to explain…") fosters the exploratory activity and the devolution of the teacher's authority to the students, as it happens in the case of open problems
- the statement requires to consider different aspects: geometrical and visual (the cube formed by little cubes), the numerical aspects (divisibility by six), symbolic (the formulas which express the number of remaining cubes and the algebraic manipulation on them). Thus the students have to use different frames and to pass from a frame to another
- it is not similar to statements proved by the students in other circumstances, thus students are stimulated to find their own way.

The setting A (secondary students grade nine)

In secondary school 18 students of grade nine (aged about 14) faced the problem in question. They had worked before in collaborative groups and thus we allowed them to work in group. The students were used to be involved in activities of exploration, production and validation of conjectures. In particular, they were able to perform these activities with the symbolic pocket calculator. The classroom was really a *community of practice*, as advocated by (Schoenfeld, 1992). The allowed time was 90 minutes. In the first 15 minutes the students were asked to work individually on the problem before starting the work in group. This splitting in two phases was decided because it happens that without an initial phase of personal reflection the interaction in the group may be only apparent and some members of the group follow passively the solving strategies proposed by their mates. During the work the teacher and the observer were at disposal of students for giving explanations and to foster the exploration. At the moment of the experiment the students did not know the algebraic manipulation of formulas, but they had used regularly the symbolic calculator; they mastered the commands *Factor* and *Expand*. The command *Factor*, indeed, has been

used before only for decomposing numbers, but it has been easy to extend this use to algebraic formulas. To have at disposal the calculator allowed keeping the focus on the problem and not on 'side issues' such as algebraic manipulation. In our intentions the resulting atmosphere in the classroom should have been rather relaxed so that the moments of strong emotions for stops or failures should have been avoided or, at least, overcome through collaboration and communication. In this situation all students, even the weak ones, had the possibility of producing some materials.

**The setting B (university students)**

The five university students participating to the experiment were attending the third year of the mathematics course. They had already passed examinations such as analysis, algebra, geometry, and topology. They worked alone and did not interact with the university lecturer and the observer. The allowed time was 60 minutes for the problem in question and another problem that we do not consider here.

FINDINGS

The teacher or the lecturer (the authors D. P. in school and F. F. in university) together with an external observer (the author M. C.) assisted to the experiment. The data were collected through the students' protocols and the observer's field notes.

Findings in the setting A (secondary students grade nine)

We had at disposal 18 protocols coming from six groups. All groups, but one, reached the solution. In Fig.1 we report the cognitive pathway towards successful proof, which emerges from the protocols.

Fig.1. Cognitive pathway towards successful proof
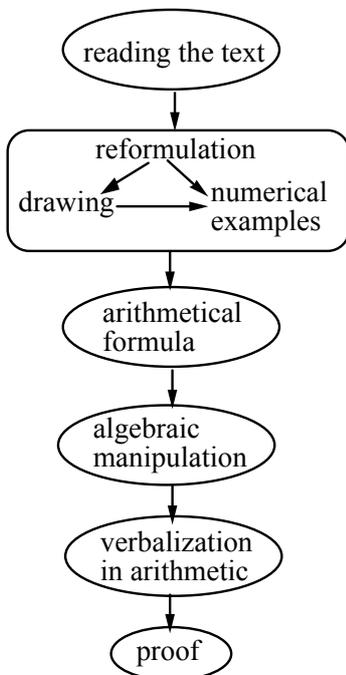
Each step of the pathway requires a shift from one frame to another. The word 'cube' in the statement pushed naturally towards the representation of a cube in the flat sheet according to empirical rules of perspective (graphical frame). Through the drawing the statement was reformulated in a more telling way. After the exploration of the drawing the students used it as a starting point for producing a few numerical examples (arithmetic frame). The drawing worked as a generic example that allowed to generalize and to produce the solving formula n^3-n. Borrowing the metaphor from (Tall & Gray, 1994) we may say that the drawing plays the role of the pivot between the particular (numerical examples) and the general (formulas). At this point the shift into the algebraic context allowed obtaining the decomposition $n(n-1)(n+1)$. The conclusion was reached by verbalizing the property that the product of three consecutive numbers is divisible by six.

To stress the importance and the peculiarity of the role plaid by the students' drawing we consider the work of a group of three boys (Andrea, Luca, Simone) in which an interesting process was produced. They started by drawing a cube. Firstly they explored a cube formed by 3^3 little cubes and went on by alternating exploration of inductive type (the cases of cubes formed by 4^3, 5^3,… little cubes) with reflections on the particular case of the cube they had drawn. The drawing acted as a generic example. The exploration of particular cases went on also after the determination of the formula n^3-n. The solving strategies were a continuous 'come and go' from consideration of concrete situations (particular cubes and calculation on them) to reasoning on formulas and attempts to write them in different ways. In this phase the teacher tried to act in the proximal development zone of Vygotskij (1978). He asked to students which ideas they were relating to divisibility. Simone mentioned multiples, Luca and Andrea decomposition. The new idea of decomposing n^3-n came through a process by abduction, see (Otte, 1997). At this point the teacher suggested using the symbolic calculator to decompose the formula. Immediately after having obtained the decomposition $x(x$-$1)(x+1)$ the students verbalized the solution: "Given three consecutive numbers at least one is even and one is divisible by three". We note that the decomposition was written exactly as we reported, that is with the variable n was named x as if the students would give us an external sign of the change from the arithmetic to the algebraic frame.

Andrea, however, was not satisfied with this solution and looked for a different process. One of the reasons of this dissatisfaction could have been the fact that the solution was found through the teacher's intervention and thus Andrea felt that he was not controlling the situation and needed to 'take possession' of the solution. He reflected on his drawing and we saw him to make gestures by hands, to think intensely until he found a new solution, based on the decomposition and composition of the original cube until a parallelepiped is obtained, see Fig.2. The teacher asked Andrea to write how he reached the new solution and why he looked for it. He wrote (for the reader convenience we have translated):

> *I was not satisfied at all with the decomposition made with the symbolic calculator (I was thinking: Why I have not suddenly thought to the factorization?)* [He is referring to the fact that before decomposing n^3-n he worked a lot around the figure] *and I was 'looking at'* [The inverted commas are in Andrea's text] *the figure, partly to see that 'monster' and partly because I wished to find a geometrical proof* [Andrea tries to give meaning to what is doing. He seems disturbed by the contamination of the geometric context of the problem with algebra]. *Rather unconsciously - may be by vent - I started to strike off the column in question. When I saw the column struck off I realized that the two remaining columns should have been moved so that a rectangle* [he means, indeed, a parallelepiped] *is formed, which is high a column less (x-1), deep equally (x), and large one column more (x+1). Since the formula which gives the volume of the rectangle* [parallelepiped] *is b•h•p, I wrote x(x-1)(x+1), which was the same to the factorization of the calculator. To better understand my idea see the sheet* [Fig.2] *with the steps of the operation.*

The expression 'to look at' suggests a way of looking guided by ways of thinking oriented to the production of a proof. The process carried out by Andrea is mainly

based on transformational reasoning. This reasoning was enhanced by three different kinds of signs used in an integrated way. We know that Peirce distinguishes among three kinds of signs: - icon, i.e. something which designates an object on the ground of its similarity to it; - index, i.e. something which designates an object pointing to it in some way; - symbol, which designates an object on the ground of some convention. Andrea uses all these kinds of signs in an integrated way. Initially the icon (drawing) is the way of paraphrasing the problem. The gestures by hands are a means to enhance transformational reasoning. In the very words of the student written in the protocol ("I would have wished to find a solution only with numbers") we see that for him symbols hide meaning, while the drawing is a carrier of meaning. We note that the student operates on his drawing in a symbolic mode. He, indeed, considers and manipulates the pieces of the cube as representatives of the algebraic symbols x, x-1, x+1.
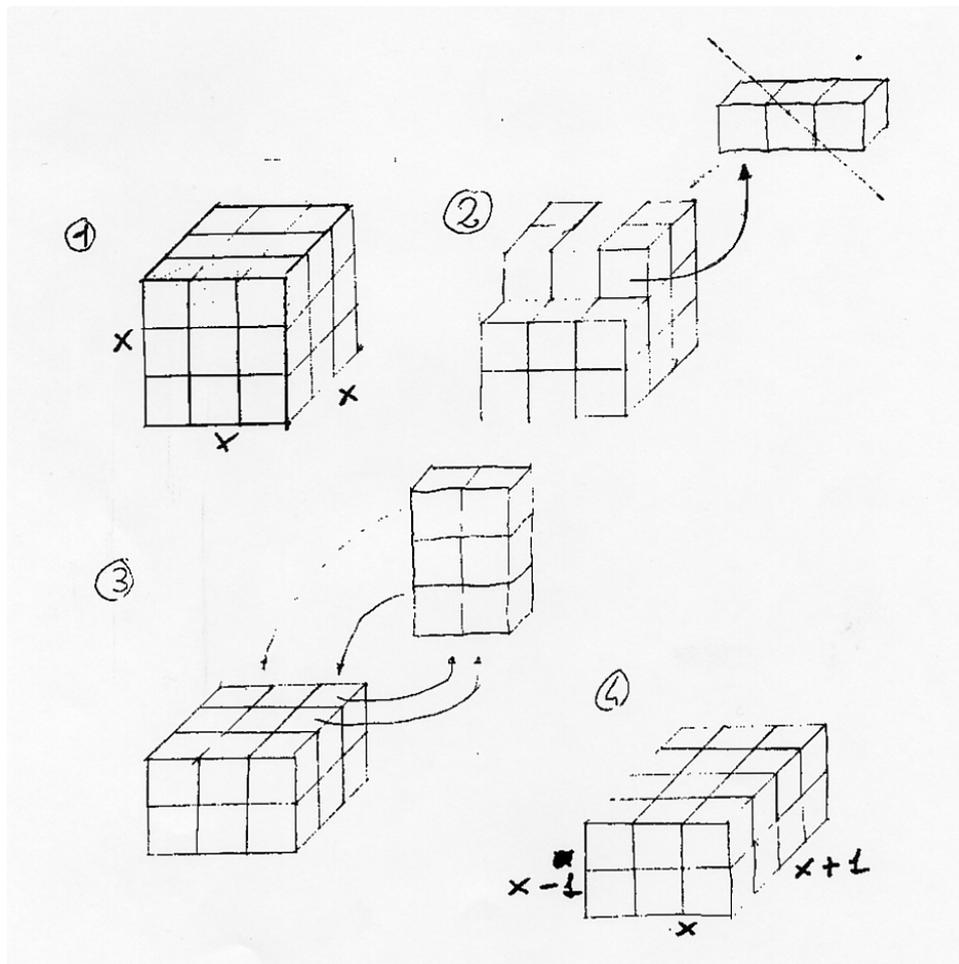

Fig.2. Andrea's geometric solution

The first mode of solution produced by the group of Andrea may be ascribed to an axiomatic-like proof scheme (they 'derives' that the number of the remaining cubes is multiple of six), while the second mode enacted by Andrea alone belongs to the transformational proof scheme (he 'sees' that that number is multiple of six). The discrepancy of schemes shown by this student is an evidence of a discrepancy between proofs which prove and proofs which explain, see (Hanna, 1990). We found

interesting that in the group the two mates of Andrea acted in a different way. They both worked only inside the algebraic frame asking for formal aspects and avoided reference to concrete situations.

The process conceived by Andrea has resemblance with the process of "cut and paste" realized by Al-Khwarizmi (1838) for solving second degree equations. In Fig.3 is reported an outline of the geometrical solution of the equation $x^2 + 10x = 39$. He starts from a square of side x, sticks on the four sides four rectangles of sides $10/4$ and x. He obtains a cross (see Fig.3) whose area is $x^2+10x=39$. Four squares of sides $10/4$ are added to the cross to obtain the final square whose area $(x+10/2)^2$ is equal to $39+4(10/4)^2$. By equalizing these quantities the usual solving formula for second degree equations follows. Al-Khwarizmi was interested only on positive solutions.
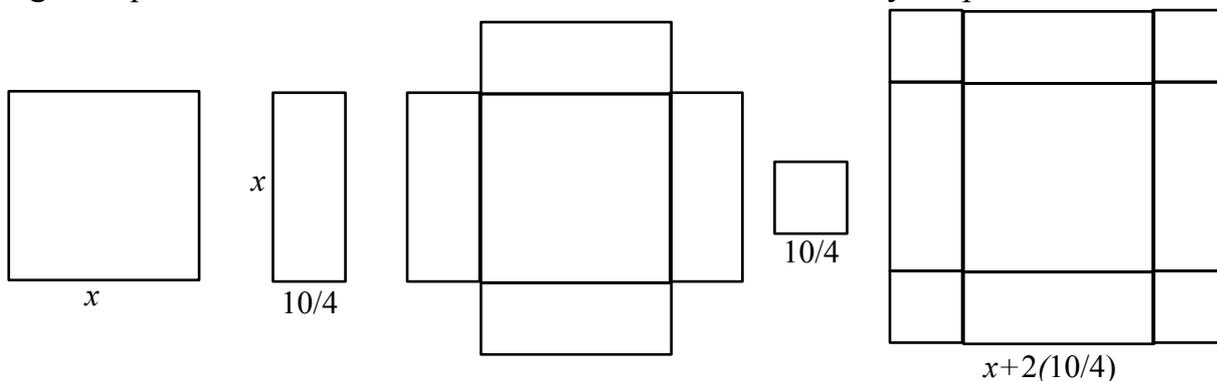

Fig.3. Al-Khwarizmi's solution of the equation $x^2 + 10x = 39$

Findings in the setting B (university students)

We add at disposal five protocols. One student produced the solution in 10 minutes writing only seven lines. He did not draw any figure: he only reformulated the statement by words introducing the variable n for the number of the little cubes in the edge and thus immediately generalizing the problem. Afterwards he wrote the solving formula n^3-n, decomposed it and through verbalization proved the divisibility by six. The rapidity of the succession of steps shows that he has a strong anticipatory thinking.

The other four students followed a different pattern (more or less the same for all). They drew a cube with three little cubes in the edge and used it very easily as a generic example to produce the solving formula without the need of exploring other cubes or numerical examples. The divisibility by six was expressed by writing $n^3-n=6q$, q being a natural number. This formula is an example of formula which has not future, that is it is not "formally operable" in the sense of (Bills & Tall, 1998, p.105) since it is not easy to use it "in creating or (meaningfully) reproducing a formal argument". One student was suddenly discouraged and stopped after just one attempt. Other students went into the tunnel of the ritual proof scheme. They acknowledged the status of real proof only to proofs that appear as the usual proofs they have seen in university courses. For this reason they did not consider verbalization as a means for proving. This pattern of reasoning is clearly evidenced in the protocol of the

student *G*. After having wrote the formula $n^3-n=6q$ she decided to prove the statement by induction. We guess that this choice was inspired by the presence of the generic number *n*. She used properly the technique of induction and arrived at a statement requiring mere arithmetical considerations. At this point, since she had 'paid her debt' to the ritual aspect of formalism, she dared use verbalization (that she refused at the beginning) to conclude the proof.

DIDACTICAL IMPLICATIONS

The university students offer materials to answer the question "Why in education more does not always mean better?" The great amount of formal mathematical knowledge and the habit to use it as the only resource for doing mathematics has inhibited the ability to look for meaning in algebraic formulas.

Our analysis of the secondary students' behavior has evidenced many aspects. Here we stress the fact that the message of the teacher had different outputs even when the conditions were the same. We owe the opportunity to grasp this fact to the style of teaching in the classroom where the experiment took place. As told before, only one in a group of three students adopted the 'cut and paste' method, his two mates preferred to look for a formal approach inside the algebraic frame. The filter of the individual's personality changes the way in which students perceive proof. The ascertainment of this fact brings to the fore the importance of studying the forms of classroom communication in relation with the different students' needs.

REFERENCES

Al-Khwarizmi: 1838, *Liber Maumeti filii Moysi alchoarismi de algebra et almuchabala incipit* (Latin translation), in G. Libri, *Histoire des Sciences Mathématiques en Italie* (253-297), Renouard, Paris.

Barbin, É.: 1988, 'La démostration mathématique: significations épistémologiques et questions didactiques', *Bulletin APMEP*, n.366, 591-620.

Bills, L. & Tall, D.: 1998, 'Operable definitions in advanced mathematics: the case of the least upper bound', in A. Olivier & K. Newstead (editors), *Proceedings of PME 22*, v.2, 104-11.

De Villiers, M.: 1996, 'Why proof in dynamic geometry?', in M. De Villiers & F. Furinghetti (editors), *Proceedings of TG 8 at ICME 8 (Proofs and proving: Why, when and how?)*, 24-42.

Furinghetti, F. & Paola, D.: 2003, 'To produce conjectures and to prove them within a dynamic geometry environment: a case study', in N.A. Pateman, B.J. Dougherty, J.T. Zilliox (editors), *Proceedings of the joint meeting PME 27 and PMENA*, v.2, 397-404.

Furinghetti, F., Olivero, F. & Paola, D.: 2001, 'Students approaching proof through conjectures: snapshots in a classroom', *International Journal of Mathematical Education in Science and Technology*, v.32, 319-335.

Gray, E. & Tall, D.: 1994, 'Duality, ambiguity and flexibility: A proceptual view of simple arithmetic', *Journal for Research in Mathematics Education*, v.26, 115-141.

Hanna, G.: 1990, 'Some pedagogical aspects of proof', *Interchange*, v.21, n.1, 6-13.

Harel, G. & Sowder, L.: 1998, 'Students' proof schemes: Results from exploratory studies' in A.H. Schoenfeld, J. Kaput & E. Dubinsky (editors), *Research in Collegiate Mathematics Education*, v.III, American Mathematical Society, Providence RI, 234-283.

Haylock, D.W.: 1987, 'A framework for assessing mathematical creativity in schoolchildren',

Educational Studies in Mathematics, v.30, 197-210.

Hersh, R.: 1993, 'Proving is convincing and explaining', *Educational Studies in Mathematics*, v.24, 389-399.

Mason, J. & Pimm, D.: 1984, 'Generic examples: seeing the general in the particular', *Educational Studies in Mathematics*, v.15, 277-289.

Otte, M.: 1997, 'Mathematics, semiotics, and the growth of social knowledge, *For the learning of mathematics*, v.17, n.1, 47-54.

Rodd, M.: 2000, 'On mathematical warrants: proof does not always warrant, and a warrant may be other than a proof', *Mathematical Thinking and Learning*, v.2, 221-244.

Schoenfeld, A.H.:1992, 'Learning to think mathematically: problem solving, metacognition and sense making in mathematics', in D.A. Grows (editor), *Handbook of research in mathematics learning and teaching*, Macmillan, New York, 334-370.

Selden, J. & Selden, A.: 1995, 'Unpacking the logic of mathematical statements', *Educational Studies in Mathematics*, v.29, 123-151.

Simon, M.A.: 1996, 'Beyond inductive and deductive reasoning: the search for a sense of knowing', *Educational Studies in Mathematics*, v.30, 197-210.

Vygotskij, L. S.: 1978, in M. Cole, V. John-Steiner, S. Scribner & E. Souberman (editors), *Mind in Society: The development of higher psychological processes*, Harvard U. P., Harvard MA.