HOW TO CHOOSE THE INDEPENDENT VARIABLE?

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A case study is presented, where the paper and pencil environment and the technological one are combined together and designed to face a subtle mathematical problem: how to choose the dependent Vs independent variables in modelling situations? We show how the combined approach allows to pose the problem in an adequate way for 9th grade students, provided the teacher interventions support suitably their learning processes. The case is analysed through two lenses from the literature: the so called instrumental approach and the notion of semiotic mediation.

INTRODUCTION

The paper presents a case study that illustrates how the combined use of technologies and paper and pencil environments can offer the teacher first the opportunity of focusing subtle but important mathematical problems not so easily accessible in only one environment, and second the tools for a positive mediation with respect to the consequent difficulties met by the students.

The "combined environment" can be thought as a tool that triggers problem posing and supports problem solving activities, provided the teacher suitably designs her/his interventions. The example we discuss here is emblematic of similar cases we met in the teaching experiments we are developing from many years with secondary school students, where the curriculum for the secondary school is "function-based" (Chazan and Yerushalmy, 2003) and developed through the combined use of new technologies (e.g. spreadsheets, DGS or CAS: see Paola, 2006) and paper and pencil environments.

The combined approach philosophy ensues from the following observations. From the one side, the students, who solve problems within technological environments, often develop practices that are significantly different from those induced by paper and pencil environments and this may offer fresh didactical opportunities:

> The curriculum with technology…changes the order and the intensity in which students meet key concepts. This change in order allows students to solve some kinds of problems that students typically might find difficult; it also either restructures points of transition between views or introduces new points of transition (Yerushalmy, 2004, p.3).

From the other side, sometimes they "naturally" use a mixed approach, where paper and pencil environment survives beside the technological one. In such cases it can be useful to exploit the didactical positive interactions of the two, suitably designing their combined use. We have observed that this methodology can be particularly useful in approaching some delicate mathematical problems, where remaining within only one environment (technological or not) may not be so productive. We shall illustrate this point showing how students choose the independent Vs dependent variables for modelling sequences of geometrical figures defined by recursive rules.

THE THEORETICAL FRAME

To properly describe our case study we use two theoretical frames in a complementary way: (1) the notion of *instrumental approach* (see Verillon & Rabardel, 1995); (2) the notion of *semiotic mediation of the teacher* (see Bartolini & Mariotti, 2008, Arzarello & Robutti, 2008).

1. *Instrumental approach.* Teaching-learning mathematics in computer environments introduces a strong instrumental dimension into the processes developed by the students. Verillon and Rabardel (1995) speak of *instrumented actions*, insofar the actions of the subjects are deeply ruled by the instrument's schemes of use (for a description of these phenomena within another theoretical frame, see Yerushalmy, 2004): e.g. to compute the roots of an equation, students can use the suitable function in the calculator modality. Instrumented actions have strong consequences on the cognitive dimensions of didactic phenomena and must be carefully considered. We shall point out how in the combined approach of paper and pencil with a specific software (TI-*n*spire) students instrumented actions contribute to modify their approach to the choice of independent Vs dependent variables in a modelling problem on recursively given sequences of geometrical figures (see below). But their instrumented actions alone are not enough to allow them to completely grasp the situation. Appropriate interventions of the teacher are necessary, as sketched in (2).
2. *Semiotic mediation of the teacher.* According to Vygotsky's conceptualization of ZPD (Vygotsky, 1978, p. 84), teaching consists in a process of enabling students' potential achievements. The teacher must provide the suitable pedagogical mediation for students' appropriation of scientific concepts (Schmittau, 2003). Within such an approach, some researchers (e.g. Bartolini & Mariotti, 2008) picture the teacher as a *semiotic mediator*, who promotes the evolution of signs in the classroom from the personal senses that the students give to them towards the scientific shared sense. We shall describe how the semiotic mediation of the teacher is crucial to support the students towards a deep understanding of the functional relationships among the variables of our problem. As a consequence, they can make an aware choice of the independent variables and draw a graph that suitably represents the situation.

THE CLASSROOM BACKGROUND AND THE TASK

The activity we shall comment concerns students attending the first year of secondary school (9^{th} grade; 14-15 years old) in Italy. They attend a scientific course with 5 classes of mathematics per week, including the use of computers with mathematical software. Since the beginning the students have the habit of working in small collaborative groups. The classroom has been chosen for experimenting a new mathematical software, TI-*n*spire (see: www.ti-nspire.com/tools/nspire/index.html) of Texas Instruments, within an international project, whose aim is to investigate the software effectiveness in mathematics learning. The students have used TI-*n*spire from the second month of school for about 2-3 hours per week. Each student has also the software at home to make her/his homework. As to the curriculum they follow, it is strongly based on the notion of function and on modelling activities through problem solving. While making the activity described below (on March 15, 2007) the students were already able to use the (first and second) finite differences techniques for analysing if and how a function grows; and to distinguish between the polynomial and exponential growing of functions or between linear and quadratic growings. For more information (in Italian) on the curriculum and these activities, see http://www.matematica.it/paola/Corso%20di%20matematica.htm.

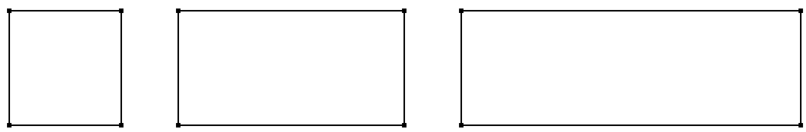
In the activity we analyse, the students, grouped in pairs, must solve a problem taken from Hershkovitz & Kieran (2001), according to the following task sheet (its working methodology is usual in the classroom).

Task

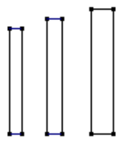
Listen carefully to the reading of the problem by the teacher. For 10 minutes think individually to the problem: do not use paper and pencil or TI-*n*spire. Produce conjectures about the change of the rectangles areas. In the successive 10 minutes discuss your conjecture with your mate; use paper and pencil only; share possible strategies to approach the problem (for validating or exploring) within TI-*n*spire. In the successive 60 minutes you can use TI-nspire to verify your conjectures, to explore the problem and eventually to solve it.

Problem

Consider the following three sequences a), b), c) of rectangles:
a) The height is constant (1 cm); the base of the first rectangle is 1 cm, while the successive rectangles are got by increasing the base 1 cm each time, as suggested by the following figures:



b) The first rectangle has height of 1 cm and base 0.1 cm; the successive ones are got increasing of 0.1 cm both the base and the height each time, as suggested by the following figures:



c) The first rectangle is a square with the side of 0.01 cm; the successive rectangles have the height always of 0.01 cm, while their bases are got each time doubling the base of the previous rectangle, as suggested by the following figures:

What can you say about the type of growing of the rectangles area in each sequence? Justify your answer."

All the pairs have produced a final document within TI-*n*spire and one of them has been videorecorded by two cameras: a fixed one for the computer screen and a second mobile one for recording the two students (L and S) while working. In the next paragraph we shall present and comment some excerpts from this videorecording. L and S are two good level achievers in mathematics.

THE SOLUTION STRATEGIES BY L AND S

In this paragraph we shall comment the strategies elaborated by L and S to solve the three questions. We shall analyse what happened only in the last two phases of their work (with paper and pencil and with TI-*n*spire). It must be observed that the classroom has been divided into two groups: one in one room with L and S and the researcher, who videorecords them but does not intervene; and the other with all the other students, who work in another room. The teacher goes back and forth from one group to the other. Hence there are long periods of time in which L and S work alone.

In phase 2, L and S do not hesitate to agree that the area in a) changes linearly. The study of the sequence b) is not so immediate. L and S build a 2 columns table, where they write the first values of the height and of the base (Table 1). L observes that the areas seem to “grow more and more” (it is the shared expression to indicate a function that increases with the concavity upwards). L wonders if this type of growing can concern all the data and not only the few considered in the table. His conjecture is that it is so provided the base does not exceed 1.

h	B
1	0.1
1.1	0.2
1.2	0.3
1.3	0.4

Table 1

H	B
1.9	1
2	1.1
2.1	1.2

Table 2

Hence he builds a second table (Table 2), which starts with the value 1 in the second column. This method is a typical strategy within paper and pencil environment; using the spreadsheet of TI-*n*spire the strategy would have been different, since students could have easily considered a lot of values and studied them with the first and second differences. At this point L generalises his conjecture saying: “It seems that it grows more and more…even because if one enlarges…it must grow more and more…two sides are always growing…hence it must grow”; and with the pencil traces in the air the “drawing” of an increasing curve with concavity upwards.

Then they pass to the sequence c). Also in this case the two students produce a table like above. At this point the teacher interacts with them and asks them what kind of growing they expect. S makes a gesture, which in their previous discussion had been used to indicate the doubling of the base. L says explicitly: “exponential…there are always powers of 2”. Then L calculates some first differences, observes that they reproduce the same values of the function and this confirms his conjecture of an exponential growing. Even if with some perplexity S accepts. Hence the students are ready to pass to the software already with many given answers. One could so expect that in TI-*n*spire they find the confirmation of their (right) conjectures. This regularly happens with the sequence a): the graphic and numeric information they get from the software are coherent each other and confirm their conjecture of a linear growing. More interesting their work for the sequence b). Once they have done the work with the spreadsheet of TI-*n*spire they wish to produce a graphic and must decide what is the independent and what the dependent variable. The second choice is obvious: it is the area. But what about the independent variable? They have some uncertainty:

L: With respect to the variation of what? Of the base?

S: Hmm…

L: Yes, L3 [he refers to the name of the variable in the spreadsheet]

S: However, it is not only the change of the base ...

L: Both are changing…both are changing… with respect to the variation of what otherwise?

S: Yes but both are changing…

After a while, the teacher recalls them that when its second differences are constant the function is quadratic and then asks them: "that this is a second degree growing, could we have foreseen it?". The students remain silent for a while; then there is the following interaction between L and T (the teacher):

L: Hence they both increase [namely height and base]

T: Before you told me that when you have thought individually you thought to the fact that to find the area you multiply the base times the height. Isn't it? You have thought to this formula…

L: Yes, hence the area could be…Then we multiply the starting number…area-one equals b times a number, b times c. The second area equals (b+1) times (c+1), hence…

L gets lost with these computations: the symbols he uses are not so good to clarify why the sequence is quadratic.

T: Of what type is the change of the base?

L: Linear as that of the height.

T: If both the base and the height grow linearly, what happens to the area?

L: The area will grow…two things that grow linearly and are multiplied…ah yes x times x!

Hence they decide that the independent variable may be indifferently either the base or the height and draw the consequent graph with TI-*n*spire: a quadratic function of the area Vs the base.

The work for the sequence c) is very interesting. The students wait for an exponential graph, but when they draw the graph area Vs base a linear function appears! The graph is so unexpected that L suggests not to consider it and eliminates it from the screen of TI-*n*spire. It is the teacher to oblige them to reconsider what has happened.

T: What about the third graph?

L: Ehmm…we do not understand, it seems that it is a linear function […]

T: What were you waiting for?

L: More and more…[i.e. a growing function with the concavity upwards]

T: The area is growing […] why?

L: Because as a base…Because we have put…also the base is changing… it changes with the *same step*.

T: Hence it is correct, isn't?

S & L: Yes, yes, yes […]

T: Be careful! We were waiting an exponential function. Namely the area were increasing exponentially, but with respect to what?

L: Of an *x* that *went on regularly…*

T: Well, what is this *x* that changes regularly? […]

L: With a *constant increment*

T: Yes but in what manner…when you have said that the area grows exponentially […] with respect to what you have thought it was increasing?...Not with respect to the base. In fact if the base grows up exponentially it is clear that the area …if the base doubles, the area doubles with respect to what?

L: With respect to what? […]

T: The area of the first rectangle is […]

L: 0.0001

T: The area of the second rectangle measures…

L: Ah, *with respect to the places*.

T: Good, with respect to the places! This problem does not appear in the preceding sequences: why?

L: Because all change with a *constant step…the base*

It is interesting to observe how the students arrive to the linearity of the graph in the dialogue (see italics) and their explanation in the notes: "…the area of the sequence grows exponentially. This appears very clear to us looking at the values of the first and second differences [of the base], which result the same as those of the area". Namely for them it is clear that linearity depends on the choice of the independent variable [the base], which in this case changes proportionally with respect to the areas. So it is clear that they do not feel the necessity of making it explicit in their notes.

CONCLUSIONS

The three questions a), b), c) are essentially solved by the students in paper and pencil environments, but at different levels of understanding. Students are pushed to enter more deeply into the relationships among the variables that model the different situations by the instrumented actions they produce. In fact, they must choose a column of the spreadsheet as independent variable to validate with the software what they are waiting for: the task is obvious in case a); problematic in case b), very difficult in case c). We call this the *problem of the independent variable.* In case b) they acknowledge that the quadratic dependence results because of the increase given to both the height and to the base of the rectangle. The reflection about the structure of the area formula (suggested by the teacher) produces L's understanding of the real nature of the quadratic law ("The area will grow…two things that grow linearly and are multiplied…ah yes *x* times *x*!"). The semiotic mediation of the teacher is based on two ingredients: (i) the necessity of passing from the signs of the spreadsheet to those of the graph environment of TI-*n*spire, which requires to explicit the two variables of the graph; (ii) the reflection on the way the multiplicative area formula incorporates twice the linear increment of the sides (bilinearity of the area function). The combined effect of these two ingredients supports the cognitive processes of L. The third case is more complex: none of the variables in the spreadsheet changes linearly with the "place". The place is a hidden variable that has supported all the previous thinking processes of the students in cases a) and b). When passing to the software, they changed the independent variable, without realising it. But while in case a) and b) the hidden variable could in some way be represented through the variables they had in the spreadsheet (case b already posed some difficulties), in case c) this is not any longer possible: it is now necessary to explicit the hidden place-variable, to see what they are waiting for. The problem could not have cropped out so "naturally" in the paper and pencil environment. Students' instrumented actions generate it in cases b) and c) but it is the intervention of the teacher to make the students aware of the problem. Its solution is crucial for developing an algebraic thinking apt to sustain the formal machinery that is necessary for modelling mathematical situations. It requires to shift from the neutral reading of the relationships among the variables of a formula (e.g. Area = base × height) to a functional reading of the same formula (e.g. Area = linear function of the base, provided height is constant, as in a). The epistemological relevance of this shifting was already pointed out by J.L. Lagrange (1879, p.15): "Algebra…is the art of determining the unknowns through functions of the known quantities, or of the quantities that are considered as known". Its didactical relevance has been stressed by many researchers, e.g. see Bergsten (2003, p.8).

Comparing what happened in our classroom with the results in Hershkowitz & Kieran (2001), we find some analogies and some differences. Our experience is more similar to what happened in their Israeli 9th grade classroom, where students "were first invited to suggest hypotheses without using the computerized tool, then to use it to check them" (*ibid.*, p. 99). In that case students could find the closed algebraic formulas for problem c), even if with some difficulties; successively they could draw the three graphs using the graphic calculator. We must observe that the focus of the problem in that experience concerned more the comparison among the relative growth of the rectangles, while in our case the attention is more on the choice of the independent Vs dependent variables. During the discussion with the teacher, the Israeli students were able to match "together representatives from different representations: the algebraic, the numerical, the graphic, and the phenomenon itself" and "the evidence provided by the different representations of the software was accepted even if, for some students, it was unexpected" (*ibid.*, p. 100). In our case the students concentrated more on the finite difference techniques and got a meaningful model of the situation; however their successive instrumented actions with the software disorientated them because of some unexpected answers, particularly in case c). In our case the software acted also as a source of problems and it has been necessary a further strong mediation of the teacher. In fact, the independent variable problem is a subtle question that has been grasped by the students because of the instrumented actions fostered by the software and of the semiotic mediation of the teacher. The two have produced a meaningful reflection on this issue and avoided that "computerized tools reduce students' need for high level algebraic activity" (*ibid.,* p.106): the instrumented actions made the question accessible to the students; the teacher fostered their thinking processes by asking them the right questions at the right moment. The use of software in this example has been complementary and not substitutive to that of paper and pencil environment. Using both has allowed to get two goals. The first one concern students learning: the dialectic between what they have foreseen in the paper and pencil environment and what they are seeing within the TI-*n*spire environment poses the problem of the independent variable and gives fuel for solving it. The second concerns the researcher in mathematics education: *combining* both environments in the teaching experiment has allowed to face the issue of the use of technologies in mathematics teaching-learning according to a fresh perspective. Our point is that the curriculum with technology "changes the order and the intensity in which students meet key concepts" not only in the "substitutive" sense that it makes "natural" different approaches to the same problem, making it easier. It changes things also in a "integrated" sense: in fact, for many reasons the paper and pencil environment continues to live in our students thinking models even if they massively use technological environments. It can be useful to *combine* didactically the two in order to pose and solve mathematical problems that could be posed and solved with more difficulty remaining only within one single environment.

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