# THE NT (NEW TECHNOLOGY) HYPOTHESIS

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## ABSTRACT

I teach a first-year undergraduate mathematics course at a business university. The course, which is part of a three-years "Degree in Economics of International Markets and New Technology", deals with those subject one would expect, such as: pre-calculus, calculus, linear algebra.

I believe that today there is a great opportunity to improve the teaching and the learning efficiency, as well as student interest, by using and letting students use a Computer Algebra System (CAS) or, more generally, mathematics software like DERIVE, MAPLE, MATHCAD, or graphic and symbolic calculator as TI-89, TI-92 Plus.

My hypothesis is that students have at their disposal all the time (during classes, while studying at home or in the University and for any assignment and examination) mathematics software with the following features:

- symbolic and floating-point manipulation
- plotting and exploring function graph
- capability of defining a function (with as many arguments as necessary)
- capability of running simple programs

Given this hypothesis (that I will call the **NT Hypothesis**), how would a mathematics course have to change? And in which way should contents, teaching of mathematical objects, problems, exercises and finally evaluation instruments be modified?

At the Creta ICTM-2 Conference, we would like to present a comprehensive description of our work, including: the project (March-July 2001), the course (September 2001-April 2002), and a first analysis of the results (May-June 2002). We decided to present at this Conference three separate papers (see also papers by G. Osimo and by F. Iozzi); each of them takes a different point of view.

## 1. Introduction

I teach a course in “General Mathematics” at Bocconi University, a school for the study of economics based in Milan, Italy. It is one of the first-year courses of a three-years program called: “Degree in Economics of International Markets and New Technology”.

The “General Mathematics” course, attended this year by 140 students, comprises 120 hours of lectures per year and deals with the following subjects:

- One-variable Calculus
- Linear algebra
- Calculus of several variables
- Unconstrained and constrained optimization
- Dynamical systems
- Financial mathematics

Every subjects covered by the course relates to various applications in the economic and financial fields.

If we want to teach a mathematics course that takes advantage of the use of technology, we can do it at two different but complementary levels:

1. By using an e-learning environment along with classical mathematics teaching: for this purpose, in my “General Mathematics” course, I used the Lotus Learning Space software, produced by Lotus.

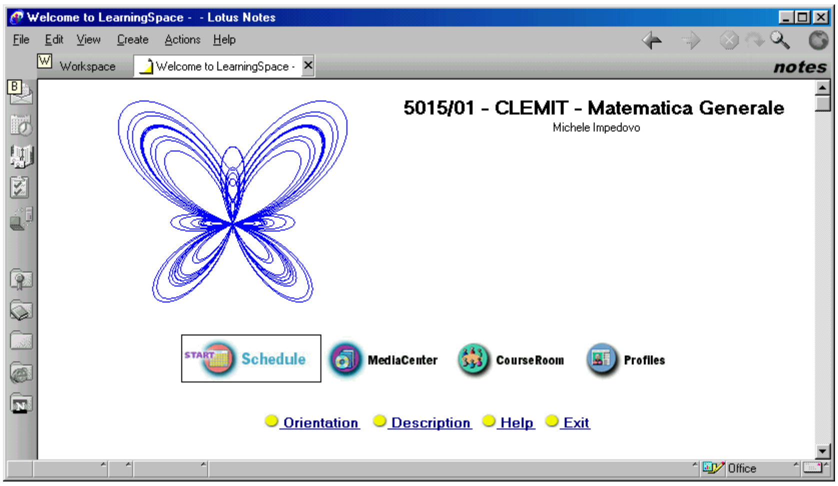



fig.1

2. By using a CAS (Computer Algebra System) software. For this purpose, I used the Mathcad software, produced by Mathsoft Inc.

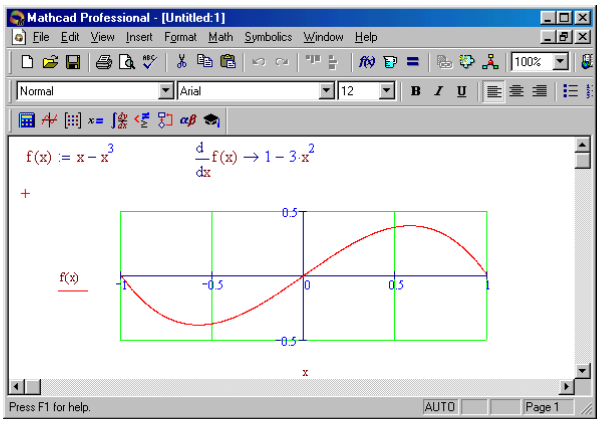



fig.2

All of Learning Space functions were used.

All information about the course was enclosed in the “Schedule” area. This included the timetable and modalities of the course; the rules for the final examination; the subjects covered by each lesson; and a list of reference papers and books for class use.

After each lesson – one lesson lasts approximately two hours – I placed two *.doc* files at my students' disposal: the first one was a 3-4 page summary of the lesson itself; the second one included a list of problems and exercises related to the subjects discussed during the lesson.

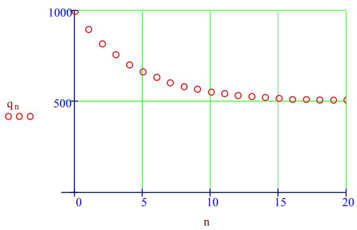
In the “Media Center” environment, I prepared for my students’ use, a set of about 50 Mathcad worksheets (and a few Excel worksheets) related to the lesson's subject.

I wanted these worksheets to play an integral role in the lessons, because on one hand they show the powerful and syntax of Mathcad, and on the other, they allow us to approach the mathematical problems from a symbolic, numeric and graphic point of view.

The “Course Room” environment is the forum in which the realization of our “computer-supported collaborative learning” project is discussed. In this space, students can post their suggestions and questions with regard to issues dealt with during the lesson. I was surprised to observe that the discussion mainly developed among the students themselves, who tried to explain to each other their own solutions to specific problems. I seldom had to participate in the forum in order to drive the discussion to the correct solution of any problem.

The “Assessment” environment was used to create exercises and simulations of examinations. In particular, in the Computer Science Laboratory, the students took two mid-term tests: a multiple-choice questionnaire and a problem that had to be solved by creating a Mathcad file. For the second part, students had to read the text of the question in the Learning Space environment, use Mathcad to build functions, calculate, plot graphics; and finally go back to Learning Space to post their solution.

Mathcad is a powerful software for numeric calculation and, only partially, for symbolic calculation. It has been used with differing functions: first of all as a “super-blackboard” - the instrument used by the teacher to show mathematical objects to the class in order to improve understanding of the concepts. For example, the following pictures represent different examples of convergence.

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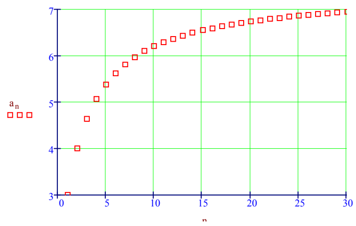


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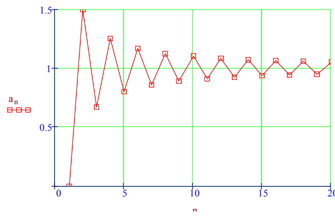



fig. 5

The students spent twelve course hours in the Computer Science Laboratory. This time was entirely dedicated to learning Mathcad syntax and analyzing its power with regard to calculation and graph-plotting.

The purpose was to provide students with an instrument for automatic calculation that they can use in every step of their learning process (during classes, training and examinations). Using this instrument, students could:

- develop both numeric and symbolic calculations using numbers, expression, functions, vectors and matrices;
- define own functions from $\mathbf{R}^n$ to $\mathbf{R}$;
- draw and analyze graphics;
- work out simple programs.

What has been described until now is what I call the *NT hypothesis* (where NT stands for New Technology): in this new context, how does a mathematics course have to change? And in which way do the contents, the teaching of mathematical objects, the problems, the exercises and finally the evaluation instruments have to be changed?

## 2. The contents

It’s obvious that some traditional skills are going to become obsolete. For example, one of the most important issues of traditional mathematics is *function analysis*: starting from the algebraic expression of a given function, we analyze its behavior and finally plot its graphic. Function analysis is the search for a graphic representation of the qualitative shape by studying its limits and derivatives. This skill becomes superfluous when students are able to obtain the graph of the inquired function using Mathcad; even more if we consider that generally it takes longer to explain how to calculate derivatives than to understand what the derivative of a function really represents. If we can save precious time during classes simply by not asking students to do a lot of calculations of derivatives and integrals, but using automatic calculation (both for numeric and symbolic calculations), we will have a lot of time left to explain their applications, instead of the calculation techniques.

Therefore, we asked students to be able to calculate in the traditional way only two patterns of derivatives and anti-derivatives: the power function $x \rightarrow x^a$ and the exponential function $x \rightarrow b^x$ (the trigonometric functions are not very interesting for economics-related subjects). For all other functions, we can use CAS.

In Linear Algebra, students have to prove their skill in using Mathcad to work with vectors and matrices (product and power of matrices, inverse matrix, rank and determinant). The idea is that students have to calculate the easiest examples in the traditional way (for instance the product, the rank and the determinant of a 2x2 matrix), while they have to apply their Mathcad knowledge to deal with more complicated problems.

For example, if a student is in front of a stochastic matrix **M** and a status vector $\mathbf{v}_{0,}$ the student must be able to explore Markov chains ($\mathbf{v}_{n+1} := \mathbf{M}\mathbf{v}_n$) without regard to their length; or must be able to solve the equilibrium equation that defines a Leontief input-output model ($\mathbf{x} = \mathbf{A}\mathbf{x}+\mathbf{d}$, where **A** and **d** are respectively the *consumption matrix* and the final demand of a productive system).

In this way, not only do we achieve our purpose of spending more time on the semantic of mathematical objects instead of on their syntax, but we provide a kind of mathematics that is:

- much more advanced, because we can investigate problems that are too complicated for traditional mathematical techniques;
- much more interesting for the students, because they are able to create their own functions and objects;
- free from rigorous scheme.

With the *NT Hypothesis,* all classical ways of studying calculus have to be reconsidered. First of all, because in a school of economics, where mathematics is mainly a tool to create patterns that help us understand specific problems related to economics, the theoretical instruments employed might be too “expensive” for the purpose. Is it really necessary to build the whole theory of limits and derivatives to find out a minimum or a maximum, when we can obtain the same result with desired precision just by exploring the graph of the function (or a rather dense table)?

I will bring an example: an exercise asked to calculate the maximum point of a profit function, given an income function R($x$) and a cost function C($x$), with respect to the sold amount $x$, defined on an interval [$a,b$]. The traditional method foresees for this case the calculation of the derivative of R($x$) – C($x$) and the search for the zeroes of this function. A student came to a different solution just by considering a table of 21 rows and two columns containing $x$ values from $a$ to $b$ with step $(b-a)/20$; the student asserted that the maximum point was the value $x^*$ corresponding to the maximum value among the values of $R(x) - C(x)$ in the table. How can we evaluate this solution?

There is another important consideration: we usually consider that a function is differentiable in every point of its domain (without discussing whether this is true or not), but in real world we come across functions defined only in some range, not differentiable or even not continuous in some points.

Let me use another reality-based example: in Italy the main tax on citizens income (called IRPEF) is a function that is continuous but not differentiable in the range $[0, +\infty)$; in fact, the tax $y$ depends on the income $x$ in the following way:

18.5% from 0 to 10,000 €

25.5% from 10,000 € to 15,000 €

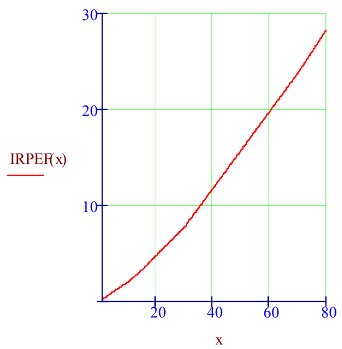
33.5% from 15,000 € to 30,000 €

39.5% from 30,000 € to 70,000 €

45.5% over 70,000 €

Using Mathcad, it is easy to define and plot the tax function IRPEF($x$) and the mean tax rate IRPEF($x$)/($x$).

$$\mathrm{IRPEF}(x) := \begin{cases} 0.2\cdot x & \text{if } 0 \le x \le 10 \\ 0.2\cdot 10 + 0.25\cdot(x-10) & \text{if } 10 < x \le 15 \\ 0.2\cdot 10 + 0.25\cdot 5 + 0.3\cdot(x-15) & \text{if } 15 < x \le 30 \\ 0.2\cdot 10 + 0.25\cdot 5 + 0.3\cdot 15 + 0.4\cdot(x-30) & \text{if } 30 < x \le 70 \\ 0.2\cdot 10 + 0.25\cdot 5 + 0.3\cdot 15 + 0.4\cdot 40 + 0.45\cdot(x-70) & \text{otherwise} \end{cases}$$



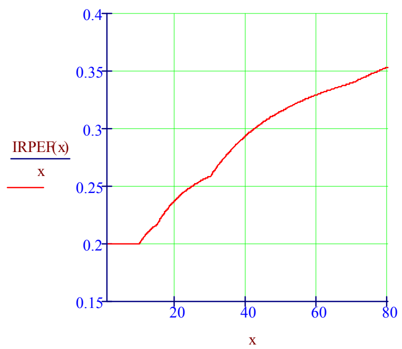




fig. 6

Recently, the new Italian government has proposed to simplify the pattern by reducing it to only two income classes, according to the following function:

23% from 0 to 100,000 €

33% over 100,000 €

Who is going to benefit from this changes? A student solved the problem simply by comparing this new function to the old one

IRPEF_2002($x$) := 0.23 if $0 \le x \le 100{,}000$

0.23*100,000 + 0.33*(x-100,000) otherwise

For these kind of functions, solving the problem in the traditional way is very "expensive", while the use of a graphic comparison is much more practical because it provides all features of the function with required precision. Other similar patterns, sometimes even not continuous, are those concerning the costs of transportation of goods, energy costs, and so on.

A very important feature that confers an aspect of innovation to this new way of teaching and learning mathematics, is the contraposition symbolic-numeric and continuous-discreet. In Italy, the symbolic and the continuous are more common. Real functions and symbolic solutions are favoured (the sequences do not appear in the secondary curriculum). For example, for the question:

$$\int_2^3 \frac{1}{x^3+1}\,dx =$$

the usual answer is

$$\frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{5\sqrt{3}}{3}\right)+\frac{1}{6}\ln\left(\frac{16}{21}\right)-\frac{\sqrt{3}}{9}\pi\,,$$

and it is very difficult to understand what the student has learned from this exercise: the symbolic solution is even more obscure than the question itself, and from a semantic point of view, it does not represent an improvement.

In this case, mathematics is only a close loop: it doesn’t have to uncover its mechanism.

## 3. Visualization of math objects and concepts

The use of a mathematical software leads to a considerable improvement in the efficiency of teaching: the visualization of calculations and graphs, the possibility to use animations and to present a huge number of examples, as well as the fact that any modifications of given parameters take effect immediately on the worksheet, make the learning process easier, faster and more efficient.

Teachers can use a “super-blackboard” that helps them and the students create something new. For example, when I introduced the concept of derivative, a worksheet like the following one was of great help:

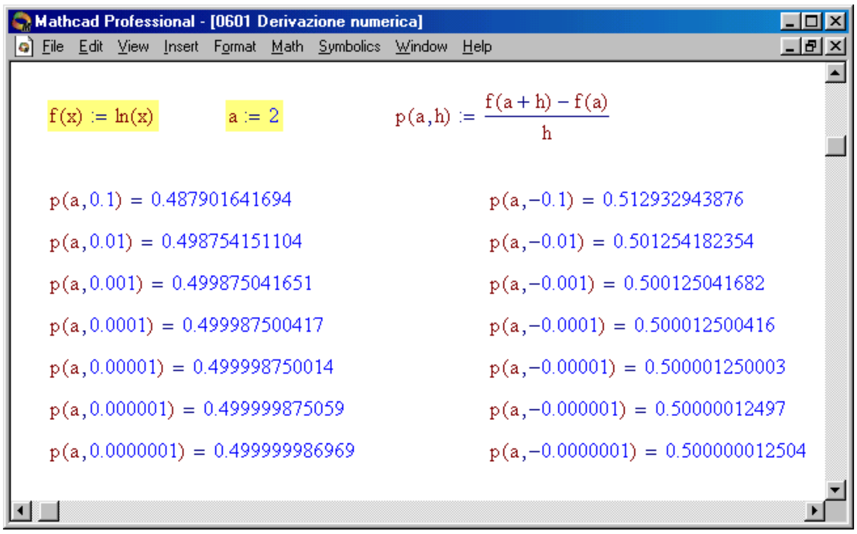



fig. 7

If we modify the function $f(x)$ and the point $a$, the whole worksheet refreshes and shows that $p(a, h)$ converges to $f'(a)$. In this way, simply by modifying the $a$ value instead of proving the symbolic expression of $f'(x)$, we promote the students' attitude to make conjectures. It takes only a few attempts to understand the trick: the derivative of $\ln(x)$ seems to be $1/x$. In the early learning phase, making conjectures is one of the most important activities: it is a signal that students have understood which problem we are trying to solve; and very often this is more important than giving a "politically correct" answer.

The opportunity to use directly in the classroom a software like Mathcad is also helpful, not only to develop mathematics knowledge, but also to simplify it. For example, let us consider the definite integral $\int_a^b f(x)\,dx$; it could be introduced with this easier definition:

$$\int_a^b f(x)\,dx \ := \left| \begin{array}{l} \Delta x := \dfrac{b-a}{n} \\ c_k \in \left[a+(k-1)\Delta x,\ a+k\Delta x\right] \\ \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \Delta x \cdot f\left(c_k\right) \end{array} \right.$$

In this case, by using an experimental analysis based on approximation, we can make the Fundamental Theorem of Calculus our goal, that is much more important than proving it.

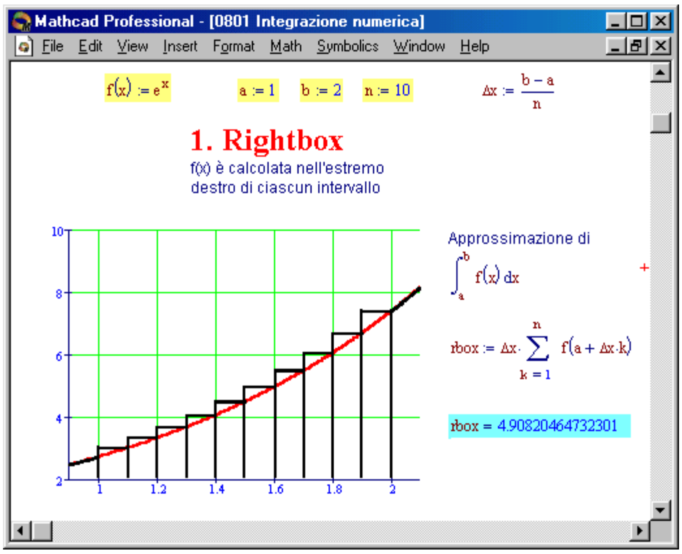



fig. 8

## 4. Problems and exercises

The *NT hypothesis* also affects the kind of exercises and problems that students have to solve. It is obvious that the students will not be asked anymore to "study" a function or to calculate on paper a derivative or an integral, but just to project the calculation so that CAS can do it for them. For example, during the final examination, students must prove that they know how to use Excel to prepare a loan amortization schedule with constant payment: the Excel worksheet must be parameterized, meaning that if only one input data (for example the rate of interest) is modified, the whole scheme changes.

Students who use Mathcad do not distinguish between a function that "can be integrated" and another one that "cannot be" (i.e. the ones that do not have an elementary function as anti-derivative), as

$$f(x) := \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

because the result provided is numerical. The complexity of the calculation is one of Mathcad's tasks. Students have to prepare some electronic worksheets in which they have set the Gauss function. To solve a problem like the following:

*The average height of a population of adult males is 174 cm with a standard deviation of 12cm. Calculate the portion of the population that is taller than 180 cm.*

students have to change only the values of the two parameters "mean" and "standard deviation".

In the same way, students must be able to calculate a least-square line using Mathcad. If we only know some points of a function, for example a demand function, and we believe that the demand decreases linearly with the price, it is very important to know what a linear regression is. (It is not really important that students are able to calculate it, but rather that they know what it represents: why do we choose as parameters the "mean" and the "standard deviation"?).

The suggested problems can be easily solved with some spirit of curiosity. During classes, when we introduced the rates at which functions grow, we presented the following problem:

*How many solutions does the equation* $x^{100} = 2^x$ *have?*

Most students plotted the graph of the function and they found what they were looking for: there are two solutions.

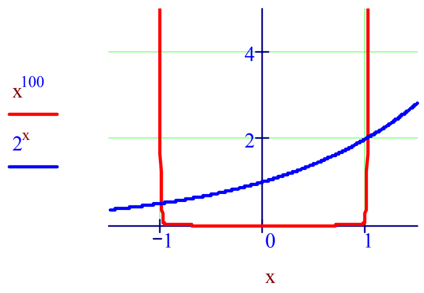



fig. 9

Some of them laboriously looked for a third solution corresponding to very high values of *x*.

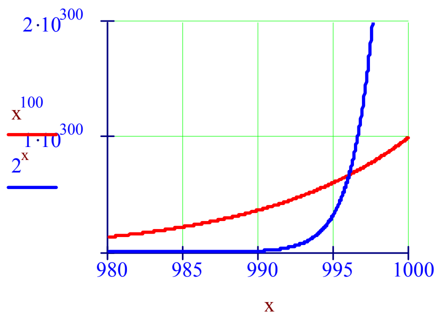



fig. 10

Only one student remembered what he had learned about logarithms, and simplified the problem.

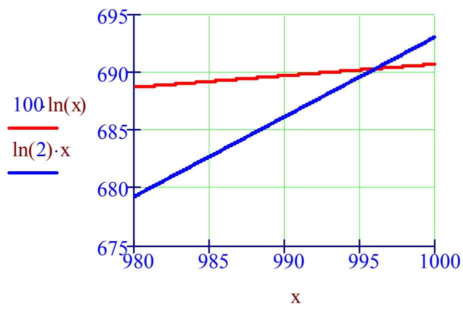



fig. 11

The one presented above is a good example of how much mathematics is mainly a matter of thinking on a particular problem: the experience is useful and enriches our ability of abstraction, but it cannot replace our capability of thinking.

**5. The assessment**

The course is divided in two semesters; each semester is divided in two further parts and we scheduled an examination at the end of each part. After these four examinations, follows a final oral test.

Students took their second and third examinations in the Computer Science Laboratory, where each student could work on an individual computer, using both the Learning Space environment and the Mathcad software. The examination lasted 1½ hours and included an eight-question multiple-choice test and one problem.

The questions were built in the Learning Space environment and the students just had to click directly on the right answer. When they finished, students had to return the questionnaire, which was automatically and immediately corrected. At the end of the examination, students could see their score in the Learning Space "Portfolio".

The problem had to be solved by creating a Mathcad worksheet and to be reported as "assignment" in the Learning Space environment. This enabled me to correct all 140 examination worksheets in a very short time, giving a judgment to complete the global evaluation, together with the questionnaire score.

To fully understand what kind of examination we proposed, here are both the questionnaire and the problem.

# General Mathematics (code 5015) CLEMIT

10 January 2002, 14.30 hours

## Second mid-term Test - Type A

1. The average value of the function $f(x) := 20(1-x)e^x$ on the interval [0, 1] is

A: $20e$ B: $20(e-1)$ C: $20(e+1)$ D:$20(e-1)$ E:$20(e-2)$

2. Loan amortization of 14,000 € in 5 equal payments at the annual interest rate i=7%. The first payment is

A: 3570 € B: 3780 € C: 3390 € D: 3420 € E: 3650 €

3. A financial operation consists of investing 2500 € with the result of cashing in 1000 € after 1 year and 2000 € after 4 years. The IRR (Internal Rate of Return) is:

A: 5.8% B: 6.1% C: 6.4% D: 6.7% E: 7%

4. The weight of a population of adult males is described by a Gauss function with average value $m$=78 kg and standard deviation $s$=12 kg. The percentage of the population weighting more than 90 kg is about

A: 4% B: 8% C: 12% D: 16% E: 20%

5. The maximum value of the function $f(x) := 50\,x\,\exp(-50x)$ on $\mathbf{R}$ is

A: 1 B: 1.23 C: 0.37 D: 0.04 E: 0.02

6. Consider the continuous function $f(x)$, positive and decreasing on the interval $[a, b]$. The function $G(x) := \int_a^x f(t)\,dt$ on the interval $[a, b]$ is

A: positive and decreasing B: increasing and convex
C: increasing and concave D: decreasing and convex
E: decreasing and concave

7. Let's consider the linear and decreasing demand function $q(p) := 480 - 10p$. The demand elasticity corresponding to the price $p_0 = 28$ is

A: −1.4 B: −0.4 C: − 1 D: −0.7 E: −2.4

8. Consider a function $f(x)$ with second order derivatives in $\mathbf{R}$. If $f(2)=3$ and $f'(2)=4$ and $f''(2)=10$ then $f(2.1)$ is approximately

A: 3.35 B: 3.4 C: 3.45 D: 3.5 E: 3.55

## PROBLEM

A transportation company has two different truck models, called A and B. The demand function of the offered service is $q(p) := 400 - 20p$, where $q$ is the amount of transported goods in tons and $p$ is the price in Euro per ton. The cost function of the company depends on which truck model is used. Using the A model the cost function is $C_A(q) := 4q + 400$, while for B it is $C_B(q) := 2q + 800$. The company always chooses the truck model that is more convenient according to the amount of goods it has to transport.

Calculate which amount is necessary to transport, in order to realize the maximum profit; which truck model should be used; the obtained profit; and the required price.

**The solution must be worked out in a Mathcad worksheet, named**

**<surname> <name>.mcd**,

*for example* **impedovo michele.mcd**

## 6. Conclusions

From an educational point of view, the innovation of the *NT hypothesis* is that students can use automatic calculation throughout all the phases of the learning process, and especially during examinations.

In front of this hypothesis we are obliged to review our knowledge and our competences as well as focus on the new skills that we want to teach our students. What kind of skills do we have to transfer to the automatic instruments? The answer is not easy (also because mathematics is only a side subject in a degree program in economics). We will have to choose one path and take our responsibility for it. A possible answer is: students must prove not only their knowledge of mathematics but also their skill in using a CAS. The automatic instrument for calculation is not just an additional one, but it is required as fundamental.

As one can see, the definition of a mathematical object (a typical sketch of an oral examination: "What does it mean that a sequence converges to the number 5?", " That it approaches 5!") becomes much more important than in traditional teaching. There is a shift: students are not requested anymore to do complicated and tedious calculations, but, therefore, they must know what a mathematical object is.

Our task is not anymore to teach students how to solve difficult algorithms; a lot of light and suitable bits can be used for this purpose.

What we have to focus on is the semantics of the concepts that we want to communicate, and we have a lot of time left for this.

The new technologies could become the paradigm of "doing mathematics". If I know what I am looking for, and I observe a syntax, I can obtain a result in a very short time. This enriches my experience, leads me to more complicated problems and, therefore, makes me more independent.

If we adopt this perspective, we can review our programs and our teaching items, and at each step, wonder what is appropriate that students to calculate using the electronic instruments. It is an extremely exciting adventure.

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