STUDENTS, MATHEMATICS, APPLICATIONS: AN ATTEMPT AT LINKING THREE DIFFERENT DOMAINS THROUGH THE COMPUTER

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Abstract. In this paper we present a didactic itinerary for students aged 16 concerning topics of algebra developed also with the use of computer. We start from theoretical problems of numbers theory and we arrive to an interesting application (cryptography).

The work described may be discussed from different point of view: here we would like to focus the most general objective underlying, that is to say the wish of the teacher (one of the authors of the paper) of enriching the students image of mathematics. With this purpose the didactic itinerary presented reproduce 'in scale' the way of working of mathematicians and involves students activities of constructing and/or interpreting formulas, conjecturing, verifying conjecture.

Introduction

In this article we refer to a teaching activity elaborated with students of 16 -17 years of age, in which many innovative elements came into play in respect to the traditional work in class (the use of the computer in the resolution of problems, the introduction of an argument not usually found in traditional mathematics courses, the discussion of an application of this argument in a real situation in every day life).

In identifying these elements and in making the relative choices various educational and cultural considerations have entered which we will discuss later. We do indeed want to underline that the basic idea that has driven our work, which is that of giving experiences to students which, using the words in (Gattegno, 1989, p. 36) make them reach awareness of the dynamics behind their actual activities. We think that the lack of this awareness is one of the principal causes of poor performance on the part of students.

The discussion reported in (Garofalo, 1989) about students' beliefs shows that mathematics is perceived of as a meaningless and rigid application of facts, rules, formulas, procedures suggested by someone (teacher textbooks) and that the acceptance of this discipline relies on authority of someone (teachers, professional mathematicians, ...).

In essence our basic aim is to give life to the mathematical experience of students: in a way to make it become their personal experience. We have dwelt on this point in other papers: for example, in (Furinghetti & Paola, 1991) we have taken as the central point the need to consider pre-existing ideas of students as a point of departure in mathematical activity- in this case calculus.

We thought in this paper that it might be useful in following the aim of which we have spoken, to stimulate in class, with the inevitable limits, the activity of the mathematician.

Thus we have tried to involve the students in a learning activity that, even if limited in time and content, reproduced the phases of the work of the professional mathematician, that we think can be schematised, at least in broad outlines, with the following diagram

Problem

Heuristic method (trials and errors, activation of various frames)

Conjectures

Verification of conjectures in particular cases

Search of a proof or a counterexample

Applications

To reach this objective we thought it useful to choose a “friendly” argument for the students; an argument that, at the same time, allowed the acquisition of knowledge and skill foreseen by the programs and particularly important for high school students.

The set of natural numbers suited our needs: in fact with natural numbers the students were having their first experiences in mathematics. Besides even in the first elements of the theory of numbers there are found problems highly significant and intellectually stimulating, but at the same time, expoundable in a very clear and simple manner.

Dealing with defined problems in the set of natural numbers also allows one to deal in a meaningful way with concepts particularly important in mathematics, for example the concept of the truth of a proposition, of conjecture, of validation of a conjecture, of proof.

Last, but not least, to work with integers allows a motivated introduction to algebra, one of the arguments in which students must reach a good technical mastery.

The choice of starting from problems in the theory of numbers (even if simple) seemed to us to be suggested also by motives of a character more strictly cultural. We are encouraged in our opinion also by works like (Dunham, 1990) and (Hardy, 1969) who give to the theory of numbers a place of particular importance in mathematics.

Description of work in class

The work has taken place in a third year of high school (*Liceo classico*, students 16 - 17 years old), which follows an experimental course in mathematics proposed by the Ministry of Education and in which one of the basic innovative elements is the use of the computer. For a fuller description of these new curricula we refer to the work of (Furinghetti, 1994).

The basic knowledge of mathematics of the students was very low: for this reason the choice of starting problems with natural numbers proved to be a happy one.

The activity was done during a term of four months (about half the scholastic year) for three hours a week. This was the program:

1. Reading and elucidation of definitions and propositions treated in textbooks of mathematics. This activity was not limited to the early lessons, but has been carried on continuously.
2. Research for formulas that give the sum of the first natural numbers, of the first even numbers, of the first odd numbers and of the square of the first natural numbers. The aim was to make the concept of formula more familiar to the students.
3. Explorations of formulas that give rise to prime numbers; formulations of conjectures; showing

proofs and counterexamples. The aim was to make the students aware that they must subject every
conjecture to a careful analysis.
4. Modular congruences and their properties. The aim was to revise the knowledge of elementary mathematics of the students such as those relative to the criterions of divisibility, using the properties of modular congruences. We hoped, by this means to further motivate the students to accept the activity of demonstration.
5. Classes of remainders $\mathbf{Z}_n$. The aim was to present to the students "new systems of numbers" and to
involve them in the exploration of the properties which are possessed by the structures { $\mathbf{Z}_n$,+}, {$\mathbf{Z}_n$, · } and by other algebraic structures.
6. Vectors and matrices as abstract structures of data. We gave some indications of operations involving scalars, vectors and matrices used in the progress of the course.
7. Cryptographic codes. The object has been to give meaningful application of the sets $\mathbf{Z}_n$. We have
followed the approach of (Childs, 1989). Cryptographic codes are not a part of the official programs, but were used to create an interesting application. They have previously been used as a teaching subject by other authors, for example (Snow, 1989)

The work was done partly in class and partly in the computer laboratory. In the laboratory there are 11 computers; the 31 students of the class work in groups of two or three.

The high number of students has been a big problem in carrying out the proposed activity in a correct and useful manner. In particular often it has been necessary to interrupt the work of some groups in order to help other students who were not managing to get on with their work, because they had not understood what was needed.

At other times, the teacher has had to begin the phase of systematisation (that is pulling together the threads of the students'work) before some groups had reached a point in the work that made the systematisation meaningful. This was in order to satisfy , before the end of the lesson, the demands for discussion of the strategies used by different groups.

Also the work in the computer laboratory, particularly at beginning, was difficult to manage, because there was no student capable of using a computer on his own.

Theoretical background

Because of the complexity of the context in which we were working (many students, new subjects, new ways of working) we had to adapt our basic ideas, influenced by constructivism, to the different situations that were thrown up in class.

One of the methods that suggested itself to us was that of inquiry approach, presented by (Borasi & Siegel, 1994). This happened particularly in the early phase of the activity. In fact we started with problems that were not well formulated and which little by little became clear through reflection on the part of the students, with the teacher functioning as co-ordinator.

The broad aims were always kept in mind, even if the point of arrival was not too well defined. For example, the discussion about an imprecise expression of Goldbach's conjecture made the teacher (whose principal aim was the understanding of a mathematical proposition) realise that the students hadn't understood correctly the idea of prime numbers. So an activity about prime numbers was begun, which brought the teacher to the proof of the conjecture (suggested by the students) that prime numbers are infinite.

The proof, which was not accepted by the students, brought about a discussion on the demonstration of the reduction to the absurd, and on its logical structures.

In the inquiry approach knowledge is seen as a construction of thinking by means of a process of inquiry motivated by ambiguities, anomalies and contradictions, and conducted within a research group. Understanding comes to be seen as a generating process of the construction of meaning, which requires either social interaction, or personal construction within the motivating situation.

Teaching is seen as an activity which creates a rich environment of motivation for inquiry and of conditions that can catch the interest of a group of students. Even if we feel particularly inclined to this pedagogical position, we have had to reject the idea of treating with such a method the major part of the activities, because of the large number of students and for the small number of hours of lessons per week (three) and also because in some cases the study of a system of rules and common syntax was required. On the contrary we wanted the students to have an acquaintance of such a system of rules and common syntax in a reasonable time. So we often opted for "learning apprenticeship", a methodology that, as has been said in (Arzarello, 1994), involves the typical aspects of the constructive theory of Brousseau, but takes into account also the aspects of support on the part of the teacher.

The apprenticeship embraces learning seen, on the part of the beginner, as imitation of the expert, and also of the support that the expert can give to the learner, by explaining his strategies, and difficulties in resolving a problem.

The beginner finds himself immersed in the climate of a workshop of the Renaissance, in which he learns by watching how it is done, doing it and continuously discussing what has been done and the aspects of the knowledge of how to do it, either with experts or other apprentices.

Adopting a terminology of (Lins, 1994), we can say that our activity is justification - driven, rather than solution - driven, but by the end of the lesson we try to have guided the students to a formalisation and systematisation of the results.

As far as the analysis of the behaviour of the students is concerned, we are interested principally in two fields of research, that of demonstration and that of algebra.

We are particularly interested in studying the production of conjectures, examples and counterexamples, and above all we want to see if the teaching situation in which the student is placed stimulates the production of conjectures, examples and counterexamples.

Regarding the field of research in demonstration, we note that, while traditionally one worked in a geometrical field, today one observes a shift of interest into other fields, in particular into that of algebra.

Regarding the research in algebra, we see instead that only recently systematic research has been begun into aspects in connection with more advanced contents in respect to that treated in traditional investigations (Dubinsky, Dautermann, Leron & Zazkis, 1994)

Description of an example of working in action

The details of the work in class are described in (Paola, to appear). Here we give an example of how the work was done, describing a lesson (the sixth, three hours). The following problem was given to the students:

" You must say if the formula $n^2 - n + 41$ give rise to

a) all the prime numbers b) only prime numbers c) also some non prime numbers.

You must fully justify every answer you give (you may use the textbook, paper and pencil and computer spreadsheet)".

The students used the spreadsheet in the construction and validation of conjectures. In this activity the concepts of formula, prime number and function entered. Besides the students through the exercise can improve their mastery of the computer spreadsheet.

The students work in groups of two or three; they can use calculating instruments. The teacher behaves like an assistant, starting off discussions within the different groups. He puts questions which should lead the students to think. To the questions of the students the teacher generally answers with other questions, having the aim of defining more precisely the problem and to lead the thoughts of the students towards the declared objective.

Sometimes, when the students are engaging in attempts that are bound to fail, the teacher behaves like a research co-ordinator, suggesting strategies, indicating opportune counterexamples. However, the attitude of the teacher is never dictatorial: he doesn't impose a strategy; he suggests it. In fact he tries to have a dialogue with the groups of students and to give them the responsibility of the final decision as regards the choice of the strategy to adopt to resolve a problem. Moreover, he tries to encourage the discussion between the students and use eventual mistakes made by them as a starting point for the following activity of analysis and systematisation of the working group.

A priori we expected that the majority of the students would have said that the formula $n^2 - n + 41$ gives rise only to prime numbers, but not all the prime numbers, justifying this answer by showing some particular examples. So it was interesting to see if the use of the computer would have allowed the identification of counterexamples and thus the pointing of some students toward the correct answer.

We observed that in the computer laboratory the students try to verify, with paper and pencil or by the aid of the computer, the formula in a few particular examples (four or five). Because of the confirmation of the validity of the formula in those particular examples almost all the students convinced themselves that the formula gave rise only to prime numbers. In particular they thought that all the prime numbers greater than 41 should be generated, and only those. There were only two students that found a counterexample (for $n = 41$).

One of those students said: " I have chosen 41 because I have seen it written in the formula and because it is logical that if you multiply a number by itself and after you subtract and add the same number, the number doesn't change and so it is a multiple of 41. The conjecture is not valid for 41 nor for multiples of 41". The other student said : " I have thought of 41, because in my opinion the creator of the formula tried with all the numbers from 1 to 40; he saw that by substituting *n* for those numbers in the formula, he always obtained prime numbers, but he didn't realise that 41 wouldn't work".

One notes also further conjectures made by other students ("neither 41, nor its multiples, neither 42 nor its multiples give rise to prime numbers"; the numbers generated by the formula finish with 1 or 3 or 7" ...) . Some of these conjectures were rejected by the use of counterexamples, others were demonstrated, others neither demonstrated nor rejected.

The use of computer

As has been discussed in the paper of (Morgan, 1994), the activity done with the aim of stimulating mathematical creativity presented considerable difficulties, because the specific aspects of the conjectures co-exist with aspects connected with the manipulation of formulas and often it is just on these aspects that teachers (and consequently students) focus their attention.

It seemed that the computer used as a mathematical assistant, could have been a real help in overcoming this difficulty.

This basic hypothesis of our work concerns either the overcoming of the manipulative phase, because the computer allows, as it is said in (Heid, 1989, p. 416) “to redirect the study of procedures from the mastery of component procedures to facility with more global superprocedures”, or the possibility of a new relationship between pupil and teacher, induced by the computer, which modifies the behaviour of the students in resolving a problem, as is said in (Artigue, 1995; Balacheff, 1991).

Later we discuss if and how these hypotheses have been realised during the work. In fact, at an early moment, in the work in class a lot of problems connected with mastery in symbolic manipulation were delegated to Derive. This allowed us to go suddenly to the heart of the problem, starting with the study of the concepts and the analysis of the procedures. In other classes in which the computer is not used one spends a lot of time in mastering symbolic manipulation.

We realised that the possibility of using an instrument of computation so powerful as Derive would encourage some students to look for resolutions of problems which they could not yet manage with confidence from a point of view of the techniques of calculation.

This behaviour was observed also by other researchers. For example (Heid, 1989, 417) writes that “students of different ages have been interested in exploring the conversion of fractions to decimals” which involved also researches into theory.

In a problem which concerned cryptographic codes, and which required the use of square matrices 2 X 2 and 3 X 3, a significant part of the students multiplied matrices, determined the inverse of a given matrix and, finally, resolved the proposed problem having only a broad idea of matrices and of vectors and having seen just once an example of product among matrices and just once an example of determination of the inverse of a given matrix.

Among the positive aspects connected with the use of the computer and observed in our teaching activity, we recall that, in using Derive, the students learned to avoid syntactical mistakes, for example they learned the correct use of brackets. In fact the use of Derive shifts the attention of the students from aspects tied to the calculation toward the syntactic correctness of expressions. With paper and pencil there are not usually questions like “Is this written correctly?”, but with the computer the students pay greater attention to the syntactic aspect.

Only the weaker students - those with more difficulties- let go the “beast of calculation” and went ahead on their own without any control (in other words, they abandoned themselves to the irresistible impulse to calculate without any checking of what they were doing).

In the second part of the work we used Derive in the study of the sets $\mathbf{Z}_n$. We made a program which reproduced the tables of addition and of multiplication in $\mathbf{Z}_n$. The students explored these tables and made some conjectures about the properties which ought characterise the tables themselves. For example, many students discovered that an element of $\mathbf{Z}_n$ has its own inverse when it is prime with *n*. Others students believed that this property could have allowed the possibility of saying that , if *n* is an even number, then all the odd elements of $\mathbf{Z}_n$ have their own inverse.

Others students rejected this conjecture by showing some counterexamples.

So among the students there were numerous exchanges of views which made the work sessions lively and stimulating. In these cases the principal function of the teacher was that of directing the traffic of the information.

As one can see, the environment of the traditional class is changed: the student becomes a protagonist and the base of the work is the communication among the students. The teacher is more a regulator than a transmitter of information.

During the course, in some groups of students, we observed very significant improvements concerning syntactic correctness of the algebraic language used and the specific language of the used software.

We think that the origin of the improvements was the need to communicate with the computer and to use it as the teacher used it. For these students, while mathematics is something that belongs to the teacher's world, the computer is a part (or must be a part) of their own world. There is then a different disposition to learn which, if it is well administered, can help in reaching significant improvements.

Besides this third pole (the computer) is like a mediator between teacher and learner and in some cases becomes the privileged interlocutor of the students that feel it neither like a judge (the role of the teacher), nor like someone who always knows the right answer, but it is seen like a peer - collaborator.

Burdens and delights of the computer. Observed behaviour of the students.

From the point of view of the learning, the fundamental subjects that have been discussed are the following: the relationship between the students and the computer, the production of conjectures, the production of formulas in algebra.

We present some significant behaviours observed in this context, underlining that many of these aspects do not depend on the specific software that has been used.

In regard to the computer, we share the opinion of (Artigue, 1995), that compared with the many enthusiastic presentations of the use of computer in class, there are relatively few critical and systematic studies about this subject.

In fact the early impression about the use of the computer in class is generally so positive that it is hard for the teacher or for the researcher to overcome this exagerated phase of enthusiasm. In this experience we observed all the advantages of the use of computer listed in (Artigue, 1995). However we want to consider also some negative aspects which lead to caution and to reflection.

Firstly we observed that some students (even if less than 20%) have had difficulties in beginning a positive dialogue with the computer, which remained, more than anything else, an unreliable and mysterious instrument, and perhaps yet an other imposition of the teacher. It is enough to think that such students checked the calculations made by Derive with their pocket calculators and there were no precise indications that such an attitude depended particularly on the difficulties with the specific syntax of the software. Besides we observed that even if the computer favours the production of conjectures, it seems to inhibit the validation of these conjectures. Many students try to justify general assertions by means of some examples given by computer.

It seems, moreover, that the activity of loocking for counterexamples is not particularly stimulated by the use of computer. We want to say that often the students, having at their disposition a very powerful instrument of computation, give up an intelligent search for counterexample (that is a search that allows the discovery of critical cases for the conjectures) and so it limits the search to a random or sequential search of all the possible examples. These behaviours can be counterproductive in a teaching course that doesn't want to lose sight of some abilities that in our opinion continue to deeply characterise mathematical activity, like that of proof itself. For example some behaviour (even if not very extensive and persistent) of the students who before the screen seem to conduct themselves as if in a dream, when the faculty of reasoning seems to have gone to sleep. Their capacity for critical analysis becomes inhibited; they accept almost all that appears on the screen without subjecting it to analysis; they also worked in a very fragmented way; they started a strategy also when the plan of working had not been thought out.

The problems of which we have spoken are typical in the affective relationships between student and computer, but there are other problems, more technical, specifically tied to the communication with the computer. In fact with the use of software programs that have their own syntax, the problems of translation from natural language to that of mathematics are sharpened, because there is another translation to do. We want to underline that a lot of students turn on their computer before reading the text of the problem. Sometimes the students give to the computer instructions in a language very close to the language which they use in their mathematical activity, without paying attention to the specific aspects of the used software.

The spreadsheet has been used at the same time with other languages and application packages with the aim of making the students think about the need to communicate with the computer, following precisely syntactic rules which vary according to the software used. The same problem has been faced by both the students and teacher with different software packages. For example the students have seen how it was possible to solve the problem of determining the greatest common divisor between two given numbers using Lotus, Turbo Pascal and Derive.

On the other hand one of the basic elements of the relationships with a computer is to understand when it is useful. More general problems have been presented to the students with the aim of convincing them of the need (in some cases) to produce proofs (which the spreadsheet cannot produce). For example some students proved that the formulas $n^2 - n + 41$ and n^2-79n + 1601 do not generate only prime numbers: some had noticed that it is enough to replace n respectively with the numbers 41 and 1601; others found only 41, while others had found 41 and 81. We asked the students if a formula of the type $n^2 - n + c$ can exist that gives rise to prime numbers only.

After we asked the students if a polynomial formula can exist that gives rise only to prime numbers. The answers to these more general problems were of the following type: “of course, because there are infinite formulas of this type and so some of them will give rise to prime numbers only”. The students did not produced anyother answers and above all they did not feel the need of prooving their assertions.

The negative answer to the question given by the teacher, and the relative proof, did not convinced them, perhaps because they found it difficult to “put in formula” and to manage the formulas and their meanings (Mac Gregor, 1993). This is a typical case in which algebraic difficulties interfere with those tied to conjecture and demonstration and became a particular difficulty in the demonstration itself.

A proposito of theese difficulties proposed we want to recall, as a further hint in all this, the words of two mathematicians [Davis & Hersh, 1986] who say that abstraction and generalisation are two ways through which meaning is transformed and sometimes lost. They also say that the computer could bring an excess of formalism within which symbols lose their meaning or become the only arbiter of their meaning. In this case we use the words of Davis and Hersh only as an inducement to thinking about a teaching that allows intelligent learning of mathematics.

We don’t go into the merit of the question in this paper, but we recall that there are studies in this direction that have already been started. For example, in the case of algebra in which the abstract and symbolic component is very significant, there is the work of (Arzarello & Bazzini & Chiappini, 1995).

Others behaviour observed during the working groups are the following:

- difficulties by the students in finding a meaning in the symbols and in the formulas used. For example, when we defined the inverse of an element in an algebraic structure, a lot of students told us that in $\{\mathbf{Z}_{26}, \cdot\}$ the inverse of 5 is the number 1/5, without thinking that 1/5 is not an element of $\{\mathbf{Z}_{26}, \cdot\}$.

- Tied to the difficulty of finding a meaning in algebraic expressions is the tendency of letting go the "beast of calculation" without any abilityt to control it. The students, in other words, favour procedural and computational aspects and do not check results. For this reason they lose completely the meaning of what they are doing. For example, when we have asked the students to determine the inverse of a given matrix having its elements in $\{\mathbf{Z}_{26}\}$, many students calculated the inverse matrix by the means of Derive and, when they obtained as the result a matrix having fractions as elements they were not worried by the fact that the obtained elements were not elements of $\{\mathbf{Z}_{26}\}$. They continued to operate with that matrix in an automatic way. In this case the use of the computer has probably sharpened the tendency of not bringing under control the "beast of calculation". We observed in the students attitudes that we may call "formal unaware attitudes" rather than "formal sophisticated attitudes" (Prodi, 1977).

- Lack of flexibility in identification in a context: for example, after doing a work about cryptographic codes in which we used some properties of the sets $\mathbf{Z}_n$, the students continuoed to stay in that context, also with problems that required working with natural numbers and not with the classes of remainders.

- Tendency to generalise properties which are valid in the set of real numbers to other algebraic structures in which they are not valid.

For example the students thought that every algebraic structure was a integral domain, like R. Besides a lot of students thought that the equation $x = x^{-1}$ when x is an element in a group has only one solution which is the identity element of the group. This case was discussed also in (Hazzan, 1994).

In particular, about the problem of finding a formula that expressed the sum of the first n natural numbers, the students meet with these difficulties:

- they did not understand the statement of the problem, also in particular cases. For example with the aim of having the sum of the first three natural numbers, two of the ten working groups added together some prime numbers; three groups did not produce any example, until helped by the teacher; two groups wrote 1+3+5 = 9 .

- The students did not understand what was ment by the "sum of the first n natural numbers". Some frequent reactions from the students are of the following types: "n can be infinite, so the sum is infinite" ; "what is n ?" ; "How can I calculate the sum if I do not know what n is it?"

There were also some interesting ideas: for example a group found the recorsive formula

$S_n = S_{n-1} + n$

while another found the formula

$S_n = n(n+1)/2$

starting from the hypotheses that the sum was a second degree polynomial. That allowed, in the phase of systematisation, to introduce a comparison between recorsive and iterative formulas.

Interesting behaviours in the activity of teaching research have been observed during an exercise in groups about linear congruencies.

The students had to resolve the two linear congruencies

$22x \equiv 19 \pmod{17}$ and $24x \equiv 18 \pmod{17}$

and they had to explain why one can say that

$2/3 \equiv 7 \pmod{11}$.

The exercise demanded, in a preliminary way, the reading of a passage of the number theory adapted from (Davenport, 1994, Childs, 1898) in which was reported the "little Fermat's theorem" (which prove that a not nil element a of $\mathbf{Z}_p$, with p prime, has a^{p-1} as its own inverse), and its demonstration. We note that the students do not use the Fermat's theorem to solve the proposed exercises given, even if the theorem itself allowed for elegant and speedily solution. Perhaps that has been causeded by the obscuring effect of demonstration. But we think that there is another hypotheses to take into consideration: that is, may be the students thought of the statement of the theorem as a point of arrival and not as a starting point that allows reaching other results.

In some cases it has been seen that the computer brought about a bypass of the rational approach, favouring visual explanations, generalisation of regularity, extrapolations of law which are valid in certain fields, and all that supported by pseudo-argumentation of the type “one can see” , “of course!”.

Finally, another aspect which we think should be investigated is whether the use of computer can increase the tendency of the students to use analogy, without having in mind a clear concept of the domain of validity of a statement or of a formula.

We may draw some preliminary conclusion from our observations: the use of the algebraic language has been useful in the activity of validation of conjecture, while the use of the software supported the activity of production of conjectures. We obtained more significant results in the activity of production of conjectures rather than in the activity of validation.

Starting points for future research in mathematics education

We think that the scenario described in this paper is particularly interesting for the study of the new environment that is created in class with the use of the computer, above all for the aspects tied to social interaction.

One of the most significant changes induced by the computer concern the exchange of information among the working groups, but the actual modality of this exchange still is not clear.

In our opinion the fundamental points of the question are the following:

- how do the students communicate? (is it only a communication of information, like in a news-bulletin, or is there also a manipulation and a structural modification of the information and, so, an activity of construction of knowledge?)
- does the computer help or inhibit the exchange of information in class?
- finally, since for a lot of students there is still a sort of non sharing of the rationality of mathematics, how much can the computer become a means of helping them to participate in this rationality?

We conclude with a return to our basic objective: to give life to the mathematical experience of students. It seemed valid to us our hypotesis of not starting with applications in order to give sense to mathematical activity. We think indeed that the computer is a crucial element which allows one to reach a meaningful level of mathematics (that of conjecture and that of resolution of problems) leaving out the more technical aspects, like computation.

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