MATHEMATICAL DISCUSSION IN CLASS ABOUT STUDENTS' CONCEPT IMAGES

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ABSTRACT

In this paper I describe an experience which is of the type mathematical discussion (Pirie & Schwarzenberger, 1988; Richards, 1991; Bartolini Bussi, Boni & Ferri,1995). Mathematical discussion offers great opportunities for learning, because it helps the construction and the sharing of domains of knowledge among students and teacher. Moreover class discussion provides students with the opportunities to reflect on their knowledge and, in this sense, it favours metacognition. This way of working has already been carried out and discussed in previous experiences, for example in (Furinghetti & Paola, 1995).

Here I present an experience in which mathematical discussions have the aim of creating links between concept images of students and concept definitions of mathematics (Tall & Vinner, 1981) which have already been studied by the students, in particular that of function. I consider either the discussion in small groups of students (Dekker, 1995) or of the whole class orchestrated by the teacher.

Résumé

Dans ce travail je décris une expérience qui concerne les discussions mathématiques discussion (Pirie & Schwarzenberger, 1988; Richards, 1991; Bartolini Bussi, Boni & Ferri,1995). Les discussions mathématiques offrent de bonnes chances pour l'apprentissage parce-que elles aident la construction et le partage des connaissances entre les étudiants et les enseignant. La discussion en classe offre aux étudiants des chances pour réflechir sur ses propres connaissances et pour cela elle favorise la métacognition. Cette méthod de travail a été déjà discouté en (Furinghetti & Paola, 1995).

Ici je présente un'expérience dans laquelle les discussions ont le but de créer des liasons entre les *concept images* des étudiants et les *concept definitions* des mathématiques (Tall & Vinner, 1981) qui ont été déjà étudiés par les étudiants, en particulier le concept de fonction. Je considère la discussion en petits groupes autant que celle orchestrée par l'enseignant avec toute la classe.

THEORETICAL FRAMEWORK

Language plays a fundamental role in the construction of mathematical concepts: pupils should be encouraged to discuss and explain the mathematics which they are doing and teachers should observe with particular interest the dynamics of mathematical discussion among students.

The term *mathematical discussion* has been explicitly introduced in literature by Pirie & Schwarzenberger (1988) and successively developed by Richards (1991). In this paper the term *mathematical discussion* is intended in the meaning of Bartolini Bussi, Boni & Ferri (1995) as «a polyphony of articulated voices about a mathematical object (a concept, a problem, a procedure), which constitute a motive in the teaching activity». Schoenfeld and others (1992) affirm that good teaching has to be founded on the disposition of teacher and students in basing the teaching/learning activity on the mutual sharing of ideas. I think that *mathematical discussions* are a useful tool to:

- give the teacher important information about the concept images of students
- help the construction and the sharing of domains of knowledge among students and teacher
- make the students aware that "mathematical ideas are susceptible to a variety of interpretations and that meaning can only in reality be shared after careful elaboration of one's own interpretation of the given context" (Hoyles, 1985)

- provide students with opportunities to reflect on their knowledge and then to favour attitudes to metacognition.

Besides, the observation of the dynamics of mathematical discussions offers to the teacher some information about students' attitudes toward the discipline, about the level of motivation to the proposed activities, about the abilities of students to control what they are doing and about the willingness of a student to collaborate with his peers.

It is necessary to say that mathematical discussions cannot be improvised: students *must learn* to socialize and to collaborate. The teacher must find the way to motivate students to social interaction. The teacher must work on two levels: the cognitive level, tied to curricular topics and the metacognitive level, which consists in developing in the students the awareness of what they are doing and why they are doing it. All this is not simple and, above all in Italy, an educational tradition regarding this doesn't exist: often teachers consider metacognitive aspects as marginal in class activities. In my opinion this makes it difficult to create an adequate "didactic space" for teaching and learning. Here I present the project and the realization of one of four experiences in which mathematical discussions have the aim of creating links between concept images of students (Tall & Vinner, 1981) and concept definitions of mathematics which have already been studied by the students (in this experience, that of function).

DIDACTIC SCENARIO: THE PROJECT

In this section I describe the project as it was presented to about twenty teachers, during a training course for teachers.

<u>Phase 0. The researcher (me) presents the project to the teacher.</u>

The researcher explains to the teacher the aims of the research and discusses with him about the modality of conducting the work in class. Also, the researcher points out that the analysis of students' protocols has two aims:

- to give the teacher some elements so that he can assess the students' understanding of fundamental topics of the curriculum
- to favour the evolution of the beliefs of the students into forms nearer to the meaning of the mathematical terms which are being discussed

<u>Phase 1. The teacher presents the project to the class.</u>

The teacher describes to the students the way of working which is that of work in small groups and encourages the students to work seriously, because this activity could improve the relations between students and mathematics. Then the teacher divides the class in small groups (three or four students in each group) following a criterion which he has previously determined. Moreover the teacher tells the students that the group work will be followed also by a researcher who shall observe their behavior and interaction.

Phase 2. Mathematical discussion in small groups.

It is subdivided into two parts:

2A. each student writes the worksheet 1 (see the appendix). In the following phase, each student can look up in the worksheet 1 the answers that he has given, but he cannot modify it. The teacher must not influence the work of the students.

2B. The teacher and the observer set out the classroom for the group work and give the students the worksheet 2 (see the appendix). The observer takes notes about the interactions students/students and teacher/students. The teacher, when he is called, participates in the work-group, limiting himself to clarifying some questions, some doubts expressed by the students.

The teacher tells the students that in the next lesson one representative from each group must show and explain the conclusion reached by the work-group. If some students do not agree with the conclusion of their group, they can participate in the discussion individually.

Phase 3. Mathematical discussion with the whole class orchestrated by the teacher.

Initially the representatives present the conclusion of their group-work. The given definitions are submitted to a critical collective analysis. The teacher moderates the intervention and guides the traffic of information. He underlines some ideas which are particularly meaningful for further developments; also, always with discretion, but in a resolute way, he tries to help students in picking out some *voices*, in the meaning of Batchin, developed in (Bartolini, Bussi & Ferri, 1995), which have emerged from the discussion. The observer takes notes of the dynamics of the discussion.

Phase 4. Setting up of the subject.

The teacher assumes a guide-role, putting together the more meaningful aspects of the previous discussion; in this phase a further mediator, that supports the teacher, intervenes: the *historic sources*. The teacher puts in evidence and brings out assonances and contrasts between the historic *voices* and the positions taken up by the various group-works. This phase has the value of an *institutionalization*: the teacher brings all his prestige into play to legitimatize the students' conceptions which are adequate for the acquisition of fresh knowledge. In a successive phase, a critical analysis of the definitions of the concept of function given in text-books (at least in that which is in use in class), may be useful.

DIDACTIC SCENARIO: THE REALIZATION

In this section I present one of the four realizations of the project. This experimentation was done in a class of liceo (students aged fifteen-sixteen). The teacher, Marina, is a colleague of the observer-researcher (me) and last year I was the teacher of the class.

The scenario, as in the Italian Comedy, has been modified on the grounds of the exigencies of the teacher, who spent very little time on the phase 1 (as you can see reading the worksheets in the appendix, which are presented in their final version as a result of agreement with the teacher) and who didn't give the necessary stress to phase 4.

The results of this behavior were:

- the number of apparent work-groups (where the students had no interest in passing on their knowledge or in explaining their ideas to the other members of the work- group) was much greater than the number of collaborative work-groups (where the students work together and are happy to do it: they feel that the success of their work depends on the interaction among the members of the group)
- The *voices* of history were not brought into the discussion as they should have been and so they didn't echo or reflect some opinions which emerged from the discussion.

As regard to the establishment of the work-groups, the criterion chosen by the teacher was that of allowing the students free to choose their chums. The teacher thought that this freedom couldn't be bad for the work-groups, because the students were accustomed to working in groups in previous years.

Worksheet 1. Expectations

This first worksheet has the aim of preparing the students for the discussion.

I thought that the students would have produced different examples and, from these, they would have reached a definition. I thought that all the following characterizations of the concept of function would have come into play:

- function as a curve or as a graph
- function as a "black box"
- function as algebraic expression
- function as particular subset of the Cartesian plane

Worksheet 1. Results

The following is a list of the main results of the analysis of the students'protocols.

- Function as algebraic expression
- Function as particular mathematical law
- Only one student used the concept of curve to explain that of function

- Misconception between the value of a function and the function
- Identification of a function with some useful operations to draw its graph. In this case what is seen is that students recall only the operative aspects of functions, that is the aspects which constituted the object of the class-works in the past. This should induce thoughts about the problem of assessment: does assessment have any sense if it is so perverse?
- Some students use the term "unknown quantity" instead of "variable". This might suggest that there is an excessive attention to computational aspects: the function becomes something to solve, something to which one must find the solution.

As a typical example I quote Gabriele D.:

«*A function is a particular relationship [...] Its resolution consists of eliminating from the domain all the elements which cannot belong to the function; after one finds the zeros and successively one calculates the sign, the limits, other points and, finally, one draws the graph*».

and Patrizia:

«*Domain, image, zeros, sign, limits, graphic of the function...every time that I see an equation like f(x) = ... or y = ..., I think of a function and I do the things which I have listed above*».

It is interesting the formulation "*every time that I see... I do"* which give the idea of automatism, of the sign which becomes signal. One student, Alessio, is aware of this risk:

«*I'm aware that my answers are not satisfying. This because, usually, if I see a formula of the type:* $f(x)=\sqrt[3]{\frac{x^2-1}{x^3-16}}$ *or any other relationships, it is more natural for me to ask myself "what must I do?" (and to think to a series of procedures to resolve the problem) instead of asking myself "what is this?"* »

Worksheet 2. Expectations

I thought that, even more than in the previous activity, here would have emerged the beliefs of students regarding the concept of function. I expected that the first two questions could have led towards mathematical discussion. On the contrary, the third question doesn't necessarily induce critical reflection, discussion and awareness.

Worksheet 2. Results

The following is a list of the main results of the analysis of the students' protocols and of the observation of the work of the groups.

- The use of examples is very limited: this is a little curious, because, usually the students ask the teacher to show a lot of examples. It is possible that the students had intended the proposed activity more as a demonstration to the teacher of what they remembered, rather than an occasion for reflecting on and discussing with their peers their own knowledge.
- The considerations tied to the use of an adequate language to speak about the concept of function are increased.

As a typical example I quote a piece of a discussion in one group.

Iro: « *a student of a middle school knows how to solve an equation* » (he intends a student aged eleven-thirteen)
Nicole: «*but if they know the equations, then they know the relationships!* »
Gabriele: « *you are foolish! We did relationships last year* » (he intends at the age of fourteen-fifteen)
Nicole: «*perhaps we want to explain too many things* »
Iro: «*if one knows what an equation is, then he knows what a ratio is...for example x = 27...* »
Gabriele: «*but if we speak about relationships, then we must introduce the concept of domain, and that of image ...* »
Iro: «*don't exaggerate! We are going to much into particulars ... I did real numbers in the middle class* » (he intended when he was thirteen years old) « *but I don't know how many people have done it* »
Nicole: «*if they know what a polynomial is, then we can say that a function is a polynomial* »
Gabriele: «*but we must to think about it more calmly...we can't do it immediately!* »

Notwithstanding this, the realization of the explanations for their peer and for the younger aged thirteen and for the teacher, were very similar.

- Even if they were accustomed to work in groups for two years, the students did not always interacte in a constructive way. In two groups I saw examples of excluding a member.
- Often some ideas that emerged in the individual test were not discussed in the work-group. This confirms the suspicion that the major part of the groups were apparent work-groups, rather than collaborative work-groups.
- The identification of the concept of function with the operations useful in drawing the graph is confirmed. The explanation constructed in the activities of the work-groups are always based on what to do with a function, never on what a function is or why the concept of function is interesting.

DISCUSSION ORCHESTRATED BY THE TEACHER

The students present the results of the work-groups. In the first fifteen minutes there was no discussion and the teacher didn't stimulate it. Only at the end of the presentation the teacher asked the students if there have been particular discussions during the activities of the work-groups, or if there was immediate agreement on the answers presented. At this point the discussion became lively. In particular the choice of adequate linguistic level for the student aged thirteen and for their peer and for the teacher was taken into account.

The teacher didn't intervene and so the discussion remained on the theme of adequate language to explain the concept of function: the discussion didn't directly concern the different images of the concept of function which had emerged in the written texts.

Here I quote a little piece of the discussion.

Eettra: «*we thought of three different ways of explanation: it seems simple, but it isn't. If I must explain a concept, I must use with everyone a language as simple as possible. I must look for the basic knowledge and then I can use it. The basic knowledge, in my opinion, is possessed also by a student of a middle school*» (she intends a student aged thirteen) «*and then an increase of knowledge must not increase the complexity of an explanation*»
Gabriele G.: «*There is some truth in it, but for a student of a middle school the basic knowledge is not solid and so the language must be simpler than the language used for a student like Elettra or for the teacher. Instead I think that the language used for Elettra and for the teacher is the same, because one cannot go further than the explanation which has been given to his peer: no one has this choice*»

Iro: «*mathematics aim at synthesis. So we can only say that both for the student of the middle school and for the teacher there must be more synthesis. For this reason we used the concept of curve: with the aim to be more synthetic. The Cartesian plane, the graphical representation are not necessary....* »
Sonia: «*I agree, but everyone must adapt his language to the person with whom he speaks. Imagine if a teacher were to speak us as if he were speaking to a colleague*!»
Nicole: «*we used a different language with the student of the middle school and with our peer; the problem is that we cannot use a more refined and precise language than that we used with our peer*!»
Sonia: «*but here it is written "explain the concept of function" I imagine that a peer of mine has put the question to me and then I try to produce some simple examples in a way that he can understand. Instead with the teacher it is different, because the teacher already knows the concept of function...* »
Iro: «*the problem is in the interpretation of the question! We must decide what the question means to us!*»
Gabriele G.: «*if someone tomorrow has to do a class-room test, he may ask a friend how one solves a function; he isn't interested in what a function is! It is like someone who asks you what a car is, but not how to drive it*»
Serena D.: «*I produce a practical example if my peer asks me how to study a function; but if he asks me what a function is, then I give the same explanation that I give to the teacher*»
Ambra: «*but it is clear! The teacher asks you to explain with the aim of finding out if you have understood; your peer asks you with the aim of trying to understand*!»
Sonia: «*the problem is that there can be some peers who may be don't know, but, in any case, the language used with them will be different from that which I can use with a student aged thirteen .. the concept of function doesn't depend on the person who is opposite you, but the language used to explain this concept must be different*»
Iro: «*in my opinion the concept of function will change in time, also for you...* »

FINAL LESSON

The teacher gave a lesson on the historical evolution of the concept of function, but she didn't bring out the connections between some students' images of the function and the characterizations of the concept in the history of mathematics. And so, the students have taken notes but they haven't had the opportunity of recognizing some ideas which they have, unawares, expressed in the previous discussions.

I underline here some *voices* which have been muffled by the final lesson:

- function as graph
- function as algebraic expression
- function as a mathematical law
- function as particular subset of the Cartesian plane
- tendency of mathematics to synthesis

CONCLUSIONS

I think that the more positive aspect which has emerged is that discussion offers meaningful elements in understanding if students have understood. The teacher is agree with this thought, even if she didn't enter into the spirit of the project and even if she still has many doubts about the systematic use of mathematical discussion as an educational tool. I think that the doubts of the teacher and her refusal to spend more time in the experience created a gap between the project and its realization. This fact opens up a great problem, which is that of interaction between the researcher, who proposes the activity, and the teacher, who realizes it, when researcher and teacher, as in this case, are different persons. As written in (Martinelli, Boero & Garuti, 1990) the teacher must be «convinced of the usefulness of investing time and effort in this kinds of projects [...] Other difficulties arise from the pressure of the school system, of parents and often of the students themselves, to obtain immediately recognizable results, executive performances, etc. ».

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APPENDIX. THE PROPOSED WORKSHEET

Worksheet 1 (individual work)

Instructions

You have to give a written answer to the following questions. You must not communicate with your fellows. You have fifteen minutes to answer. If you don't succeed in answering some of the questions, you must explain the reason for your difficulties. When the fifteen minutes are finished, you must not change your answers.

Questions

1. Write everything that you think characterizes your knowledge of functions
2. Give some examples of functions
3. Give some examples which are not functions (and explain why they are not functions)

Worksheet 2. (group-work)

Instructions

The following questions have the aim of starting a discussion. You must discuss in groups and try to reach agreement on the answers. Make notes of the points in your debate that you think are most meaningful, also those points which you rejected and indicate why. If you have not succeeded in reaching agreement on the answers, then you must specify the points of disagreement.

Questions

1. If a student aged twelve or thirteen asks me what is the meaning of the term *function* in mathematics, then I could tell him...
2. If a fellow- student of my class asks me what is the meaning of the term *function* in mathematics, then I could tell him...
3. If my teacher asks me what is the meaning of term *function* in mathematics, then I could tell him...