MATHEMATICAL OBJECTS AND PROOFS WITHIN TECHNOLOGICAL ENVIRONMENTS: AN EMBODIED ANALYSIS

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Abstract. *The paper faces an approach to Calculus in secondary schools within technological environments. It illustrates a case study, where the concept of function is approached in the 9-th grade using a sensor for motion connected to a calculator. Pupils can move and see the Cartesian representation of their movement produced by the sensor in real time. It analyses some excerpts of the discussion in the class after that students have tried to reproduce with their movement the graphics drawn at the blackboard by the teacher. The analysis uses a Vygotskian approach and the tools of embodied cognition to interpret the situation: a theoretical model is sketched, which stresses the embodied components. Some open problems are pointed out in the end.*

1. Introduction. The paper analyses (a part of) an ongoing teaching experiment conducted from the first years of secondary school (9^{th} grade up), where Calculus is approached early within different *experience fields* (Boero, 1995). Even if the experiment concerns all the basic subjects in Calculus (limits, derivative, integrals, as required by the Italian curriculum in the Liceo Scientifico), since our space is limited we shall illustrate only the approach to the function concept within the experience field of pupils' motion and shall discuss some problems that one meets approaching the theorems in Calculus. The experimental part of our study is an example of *research for innovation* (Arzarello & Bartolini Bussi, 1998), in which action in the classroom is both a means and a result of progressive knowledge of classroom processes.

2. The theoretical framework. Using instruments is crucial in teaching-learning activities because it can support and enhance learning abilities, putting forward the different aspects after which a mathematical object can be looked at. For example, a symbolic-graphic software can use both visualisation and mathematical symbols. Since *"vast tracts of the brain are engaged in perception and construction of imagery, [but] there are also huge areas of the cortex that are plastic and useable for a variety of activities including the many processes involved in thinking mathematically"* (Tall, 2000) such a representation *"can provide a powerful environment for doing mathematics and, with suitable guidance, to gain conceptual insight into mathematical ideas"*. In fact symbols can be used as cognitive *"pivots between concepts for thinking about mathematics"* (Tall, ibid.). The dynamic of such a conceptualisation can be described within a Vygotskian frame, since it represents a transition from the immediate intellectual processes to the operations mediated by signs and illustrates the dialectic between *everyday* and *scientific* concepts.

To investigate the specificity of such a dialectic within our teaching experiment, namely to describe the crucial cognitive aspects of pupils' learning processes in interaction with technological tools, we use three analysis tools: (i) the *embodied cognition* approach by Lakoff & Núñez (2000) (see also: Arzarello, 2000; Arzarello et al. 2002; Ferrara, this CERME); (ii) the *instrumental analysis* by Rabardel (1995) and others (Artigue, 2001; Lagrange, 2000; Verillon & Rabardel, 1995); (iii) the definition of *concept* given by G.Vergnaud (1990)[1], in particular the notion of *operating invariant*.

The embodied approach reveals crucial means for describing pupils' cognitive evolution within technological environments and for designing suitable teaching experiments. It shows a basic unity from the very perceptions, gestures, actions to the most theoretical aspects within a systemic relationship. Specifically, in the operational invariants of the mathematical concepts built up by students in our experiment, are evident the traces of their actions. For two examples in this direction and in different environments, see Arzarello (2000) and Bartolini Bussi et al. (1999). Embodied cognition is also useful to analyse the dynamics of the social construction of knowledge by pupils: specifically the metaphors, introduced by students in a group or class discussion, or by the teacher when (s)he wants the students to concentrate on a particular, possibly new, concept reveal powerful tools for sharing new ideas.

The analysis by Rabardel explores the interactions among students, mathematical concepts and technologies at school. It considers the way in which the technological tools act on the mathematical concepts and the way by which such concepts can model the didactic transposition (instrumentation process). It pinpoints that an artefact can become an instrument for a student through the appropriation of its schemes of use (instrumental genesis). The operating invariants in conceptualisation have traces of their actions as schemes of use of the artefacts: gestures, metaphors of their activity with them are mirrored in their conceptual elaboration.

We think it is possible to integrate the instrumental approach with the new results by cognitive science, in particular embodied cognition. These two approaches help us to analyse the students' activities from a new point of view. In fact, if the instrumental approach can give us the tools to analyse the use of technologies by students, in terms of schemes of use, it is not sufficient to support the interpretation of their mental activities, especially for the conceptualisation. On the other hand, cognitive sciences are perfectly aimed to study pupils' mental activities; however, their approach to conceptualisation processes in mathematics focuses *some* fundamental aspects but

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[1] According to Vergnaud, a concept consists of:

- a *reference* ("l'ensemble des situations qui donnent du sens au concept");
- *operating invariants* (invariants opératoires: they allow the subject to rule the relationship between the reality and the practical and theoretical knowledge about that);
- external representations (language, gestures, symbols,…).

does not give reason of *all* the theoretical and symbolic features of the mathematical thought. Hence we find it useful to embed our analysis within the framework of Vergnaud's definition of concept.

In short, our studies are aimed to an integrate analysis of students' cognitive processes while doing activities in which they use artefacts (a sensor and a calculator) and transform them into tools, through appropriate schemes of use, hence building the concept of function because of their active interaction with such signs as Cartesian graphs and numerical tables. In this paper we will concentrate mainly on the first aspect (embodied approach).

3. The teaching experiment. Our teaching experiment's main goal consists in introducing some basic concepts of Calculus starting with young pupils (14-15 y.); the experiment is pursued within different *fields of experience*, some of which use the support of suitable technological environments. As a research for innovation in the class, it has also some research goals, the most important of which is analysing such mediation phenomena through the lens of embodied cognition. Namely we wish to see how the cognitive tools put forward by Lakoff & Núñez can allow to understand the learning processes of pupils who interact with a technological artefact (e.g. a CBR), used to support their building of mathematical concepts. We limit to give an example of a 9-th grade class, where we have designed a dynamic approach to the function concept based on the following points: a) the newtonian idea of quantities which change in time, focusing on the first and second variations of the dependent variable; b) functions as modelling tools of concrete situations; c) using graphic-symbolic calculators and movement sensors as mediators in the teaching-learning of the function concept; d) using different social interactions in the class (working and discussing freely in small groups; general discussion orchestrated by the teacher in the whole class; see Bartolini Bussi, 1996); e) taking into consideration the affective and emotional aspects of learning mathematics: a friendly environment is created, where evaluation is put forward in not a stressing fashion.

We illustrate pupils' learning processes analysing some protocols of the teaching experiment through the theoretical tools sketched in Section 2; a particular attention is given to pupils' gestures, to exploit their embodied approach to concepts. We analyze an activity carried out a couple of months after the beginning of the teaching experiment: a pupil in each students group (3 persons each) must move w.r.t. the sensor so that the calculator reproduces a graphic equal as much as possible to the one drawn at the blackboard by the teacher. The pupils observe their mate's movement and comment her/his possible mistakes, pointing out the reasons why there are differences between the graphic at the blackboard and that produced by the calculator. We have many recorded videos, but for reasons of space here we limit ourselves to comment some protocols of a group, where a student (St1) tries to reproduce through his movement the graphic sketched by the teacher at the blackboard (fig. 1) and his mates discuss what he has done. While running, St1 looks at the blackboard and at the screen to co-ordinate suitably his movements. Moreover a mate (St2) comments St1's movement with expressive gestures of his hands.

St2: "*That is first slow* [he moves his right hand horizontally towards right], *then fast* [he hands up his right hand very fast], *then down fast* [he hands down his hand fast towards link], *then slows down* [he moves his hand towards link describing a concave descending curve in the air], down his hand fast towards link], *then slows down* [he moves his hand towards link describing a concave descending curve in the air], then fast again [again his hand up to the right]...*then it stops* [he moves his hand towards right horizontally]". St2's gestures show clearly that he has understood both the movement and the graphic. His hand gestures incorporate in a *compressed* way (Tall, 2002) the features of the time law. His hand gestures incorporate in a *compressed* way (Tall, 2002) the features of the time law. In fact when the speed is increasing, his hand moves faster, and when the speeds decreases his hand moves slower. In a Cartesian graphic the information concerning the function variation and its derivative is coded in a unique sign (i. e. the graphic) and as such it is not accessible to all. The movement of St2's hand has two aspects: the first (namely the trajectory of his hand) expresses how the function varies (the time law form); the second (his hand speed) incorporates the velocity of the moving body. This double embodiment of information is not a coding into an unknown language; it is a 'natural' representation of the movement. In fact, St2's gestures are more direct representations than the blackboard graph (i.e. a Cartesian plane with different quantities on the two axes): they are a mediating tool for grasping the situation in a more feasible way (no transcoding is needed, apart the embodied one). In a certain sense, St2's intervention represents an intermediate level between the external movement and the time law, (i.e. through the Cartesian graph), which is useful to start a comprehension process of the scientific features of the motion. It represents a stage towards the interiorisation of this scientific meaning for St2, but it also creates a possible space of communication for the class, which was not evident before. Eventually this allows pupils' evolution towards the scientific meaning of the sign introduced by the teacher at the blackboard. In factSt2's words and gestures are taken again by other students in the class: most of them use the same type of gestures than St2 while discussing the problem.

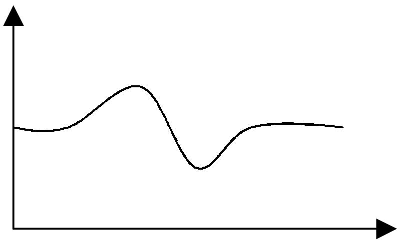

Fig. 1

Another important issue to point out consists in the role of the teacher. His interventions are crucial in order to make pupils conscious of the scientific concepts they are learning. Their experience lives still in everyday concepts, but their gestures incorporate the scientific aspects; they are typically in a 'zone of proximal development' (in the sense of Vygotsky): the teacher supports them linguistically to transcode their conceptualisation into the scientific language. In the discussion, the scientific words suggested by the teacher give a name to the gestural description used by pupils to describe the situation: they repeat the words and make the same gestures. In this blending of representations they conceptualise in a conscious, intentional and willing way, namely they conceptualise a scientific idea according to Vygotsky. However the blending of gestures and words they use show that their conceptualisation embody the traces of their actions.

Two more observations seem important. First, in the usual teaching activities for Calculus, functions, (first and second) derivatives are introduced with split meanings, generally in different moments of the curriculum. This inhibits important links between them and also any possible *meaning compression* (Tall, 2002). Using sensors allows to approach all the aspects within the same experience field. This can support the emerging of a *cognitive unit*, in the sense of Tall (2000).

Second, the social interaction seen in the class is a typical *vygotskian* situation where there are signs (the two diagrams: at the blackboard and on the screen), actions (St1's movement, St2's gestures) and their interpretation: the pupils who mimic St2 enter in a common interpretative process which allows them to grasp the signs' meaning. Through that, the meaning is interiorised and gestures become a communication/interpretation tool for successive similar situations. In particular such a tool does not block the lowest students, who would encounter difficulties using a more formal language.

4. Open problems: a dramatic gap? The example illustrates emblematically that a technological artefact can support effectively students in building meaningfully various advanced mathematical *concepts of Calculus*, e.g. functions, derivatives, integrals (see the proposal of O. Robutti at this same CERME group). In fact, a suitable coaching of the situation by the teacher makes accessible to the students the *cognitive roots* (Tall, 2002) of delicate concepts in Calculus in an embodied way; this is the very basis which can support the transition from the perceptual side to the theoretical one. However things change when one passes from the mathematical objects to the proofs even of the simplest theorems in Calculus. In this case, it is necessary to take into account the conceptual discontinuities which feature reasoning with such concepts. We shall discuss sketchily this issue, which points out some crucial didactical points, which require new investigations to know how an embodied approach can be useful to overcome the new discontinuities.

In other papers (Arzarello, 2000; Olivero et al., 2001; Arzarello et al., 2002) one of the authors has analysed the dynamic geometric software as a support for the learning of proofs in Geometry. The studies have pointed out the *cognitive continuity* (Garuti et al, 1996) which features the transition from the exploring, conjecturing, arguing phase to the proving one. That is in learning environments which can support them in their observations and explorations, pupils systematically build up the proofs of their conjectures using facts, ideas, words and sentences already produced during the exploration and conjecturing phase. It is not always a one to one translation, since some crucial restructuring processes may happen (see Arzarello, 2000), but the main ingredients remain the same. Such a is opposite to the *epistemological gap* which is observed between the informal and the formal phases of proving (Arzarello et al., 1998).

We distinguish between a *conceptual continuity*, when the ingredients remain the same within a reference theory, but the way they are arranged together may change and *a structural continuity*, when also the structure after which the ingredients of the proof are put together does not change essentially. Such a terminology comes from E. Pedemonte (2002), who defines *reference system continuity* the former and *structural continuity* the latter. While in some geometric situations the structural continuity is preserved, it is seems not always so in the case of more advanced arguments, e.g. in Calculus,: in fact, the necessity of restructuring is an index of the difficulty of the task. Context relevance is not surprising: the conjecturing and proving activities are linked with the meaning of the mathematical objects they involve as well as with the mutual relationships among them. Hence the existence and the nature of a possible cognitive continuity depends strongly on the mathematical context within which the proof is developed. Let us make a concrete example, comparing Geometry with Calculus. In Calculus things as $y=\log(x)$, $y=x\cdot\sin(1/x)$, the tangent to $y = x^3$, a continuous function... are not so immediate as a triangle in Geometry; they exist as an elaborated product of mathematics (generally incorporated by formulas). Links with perception are problematic (e.g. through a picture of the continuum or through orders of magnitude): a theoretical frame is necessary to guarantee some *ideal perception* through the mind's eye. Proofs are made of words and calculations, only apparently similar to the algebraic ones: in fact logical complexity increases, with the use of $\forall\exists\forall$ formulas; besides, calculations concern inequalities, nested within such alternated quantifications and so their status changes deeply from the algebraic one. Last but non least, the reference to drawings (figures) as generic objects is very problematic in technological environments. While with a dynamic geometry software one can give an idea of a generic triangle, the notion of a generic continuous function is beyond the capabilities of nowadays technology, as far as we know. Even in the case of arguments and proofs there are crucial differences between elementary Geometry and Calculus: the latter has more obstacles and traps for the learner. To be concrete, let us consider two different proving strategies, which are a possible cause of breaking within the structural continuity between arguing and proving phases: the first consists in arguments and proofs which require the limit definition in analysis; the second is the use of abductive[2] strategies in geometry during the exploring and conjecturing phase as a prelude to the proving one (the latter have been investigated in Arzarello et al., 1998 and 2000; Arzarello, 2000). Both strategies need a sort of mental *somersault*; however this has different features: while abductive one does not break the cognitive unity (Arzarello et al., 1998), the cognitive continuity of

[2] The following example (Peirce, 1960, p.372) is illuminating about abduction. Suppose I know that a certain bag is plenty of white beans. Consider the sentences: A) these beans are white; B) the beans of that bag are white; C) these beans are from that bag. A deduction is a concatenation of the form: B and C, hence A; an abduction is: A and B, hence C (Peirce called hypothesis the abduction). An induction is: A and C, hence B. For more details, see Magnani (2001) and Arzarello et al. (2000).

processes seems not always preserved with limits, because of cognitive and epistemological reasons. Let us see better why.

In the limit definition the somersault consists: (i) in the consideration of the $x \rightarrow y$ subordinate variations (x variations are seen as causes of y variations) during the conjecturing or in the early proof construction phases; (ii) in the deductive enchaining phase the y-universal quantification drives the x-existential/universal quantification. A shift from $x \rightarrow f(x)$ *direct* reasoning to *inverse*, $\forall\exists\forall$-reasoning is needed. Hence, we have a double inversion: from $x \rightarrow y$ to $y \rightarrow x$ and from $\forall\exists$- to $\forall\exists\forall$-quantifiers.

In abduction within geometric explorations the sommersault consists in a switching:

(i) in the modality of control of the subject with respect to the geometrical figures: ascending vs/ descending[3]; **(ii)** in the way the subject sees the mathematical objects, with respect to what is considered as given and what is supposed to be found: such a relationship usually change many times during the exploration (Arzarello et al., 1998 and 2002). In Calculus the somersaults involve also the genesis of some basic structured mathematical objects necessary in that field: the natural (metaphorical and possibly perceptual) ways after which such concepts are built up require deep somersaults to get a rigorous mathematical definition. Since the cognitive genesis of mathematical objects unlikely can avoid the 'natural' metaphorical way, this poses serious didactical questions from the very beginning of the teaching of Calculus. The problem is hard since at the moment it seems difficult finding genuine cognitive roots or other 'natural' cognitive entities which can support a cognitive continuity while proving the theorems in Calculus. Neither suitable mediating tools (e.g. software) seem to exist that can support the apprenticeship to the proof in Calculus through its structural continuity breaks, pointed out above. As we have already told, somersaults present a double inversion (from $x \rightarrow y$ to $y \rightarrow x$ and from $\forall\exists$- to $\forall\exists\forall$-formulas), which seems very hard to overcome. In particular, we have not found any evident *natural* (e.g. embodied) example of such somersaults: do they exist? At the moment the answer seems negative ([4]) and two ways remain available for approaching proofs in Calculus.

(i) A first solution consists in developing a suitable *didactical engineering*, namely designing learning environments and situations centred on such conceptual discontinuities. For example, it is possible to coach an evolution in the class from the natural, embodied roots of the mathematical objects of Calculus (like in our example or in Tall 2002) to the culture of theorems described by Boero. This can be achieved through a *cognitive apprenticeship* (Arzarello et al., 1993), where pupils are nurtured to the somersaults (i.e. to the structural continuity breaks). The role of the teacher is crucial in such an apprenticeship, insofar (s)he is responsible for the transition to the

[3] *Ascending processes* are from the drawings to the subject, who explores the situation, looking for regularities, invariants, etc. with an open mind; *descending processes* are from the subject to drawings, in order to validate or refute conjectures, to check properties, etc., which the subject has already in her/his mind (Arzarello & al., 2002).

[4] The best example of a 'natural' somersault that I know is the following (Lolli, 1992): "For each criminal, it will arrive the moment, when he will be obliged to pay for his faults"

socially shared level of mathematics which incorporates the somersaults in its historic evolution. During the exposition we shall present an example.

(ii) Another solution consists in changing technically the classic approach to Calculus and avoiding systematically the conceptual somersaults. For example, there are teaching experiments based on non-standard analysis (see the survey in Maschietto, 2002), or even books where a big ingenuity has been used to build up a Calculus which avoids Weierstrass' traps.

A major issue consists in finding suitable cognitive roots for the new gaps; our first results suggest that an embodied approach can be the right way also in this case.

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